VMS Closure Math Primer

Built from selected visuals + minimal variational geometry from James Bisgard, “Mountain Passes and Saddle Points”, SIAM Review 57(2), 2015.

Scope: This primer is not a full mountain-pass theorem lecture. It’s a picture-first explanation of: (i) optional paths, (ii) why bottlenecks/transition states are forced, and (iii) what “closure” means as an invariant under legal refinements.

1) Optional paths exist, but bottlenecks are forced

Think of a landscape where height = “cost” (action / inconsistency / residual). You can draw many different paths from one basin to another — those are your optional paths. But if the

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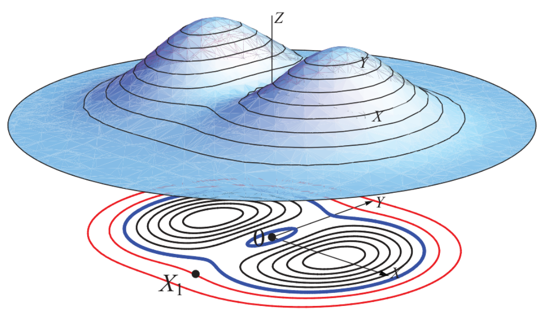


Fig. 3.1 *Mountain pass geometry: red denotes levels lower than the valley centered at* **0** *between the two peaks, while blue is a level higher than the middle. Notice that every path that begins at* **0** *and ends at* $\mathbf{x}_1$ *passes through blue!*

3. Mountain Passes. Throughout this section, we will assume that $F : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable. Suppose now that F satisfies the following:

(MP1) $F(\mathbf{0}) = 0$.
(MP2) There is an $r > 0$ and an $\alpha > 0$ such that $F(\mathbf{x}) \geq \alpha$ for all $\mathbf{x}$ with $\|\mathbf{x}\| = r$.
(MP3) There is an $\mathbf{x}_1$ such that $\|\mathbf{x}_1\| > r$ and $F(\mathbf{x}_1) \leq 0$.

Geometrically, we think of **0** as lying in a valley, surrounded by a range of mountains whose minimum height at distance r from **0** is at least α (see Figure 3.1). Ideally, there should be a mountain pass over the mountains. Unfortunately, as F in the last section shows, this may not be true. However, we can show the following.

Theorem 3.1. *Suppose F satisfies* (MP1-3). *Then there is a (PS)-sequence* $\mathbf{x}_n$ *such that* $F(\mathbf{x}_n) \to c$, *where* $c \geq \alpha$.

In the proof, we will pick a path γ that connects **0** and $\mathbf{x}_1$, thinking of it as a long rubber string. Next, we move all the points on this string “downhill” (decreasing the value of F at all the points on γ). As we do this, the string will move and stretch over the landscape. Notice, however, that since all points on the string are moved downhill, points that start in the valley will remain in the valley and points that are close to $\mathbf{x}_1$ must remain outside the valley. Thus, every time we move the string, it must still

basins are separated by a ridge, every path must cross a higher level somewhere.

Bisgard Fig. 3.1 (p. 279): mountain-pass geometry.

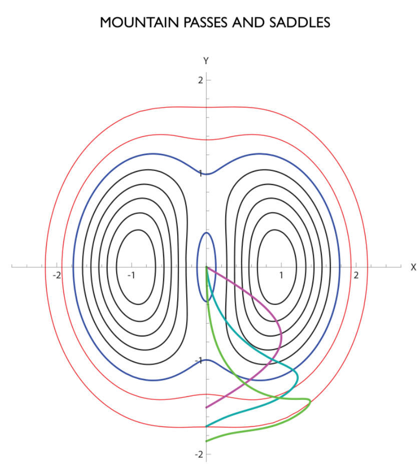
Quoted caption (Bisgard, Fig. 3.1): "...every path that begins at 0 and ends at x1 passes through blue!"

VMS closure intuition: closure isn't 'pick a path'. Closure is the statement that if you want to change basin, you pay a geometric price somewhere — a forced crossing. That crossing is the transition region your analysis has to account for.

§1 → "This is the geometry behind F0013 wave–particle boundary and the admissibility condition in PM §1."

2) Refinement deforms paths (rubber-string picture)

Now imagine each candidate path is a rubber string laid across the landscape. If you push every point of the string downhill, the path will slide and reshape — but it cannot teleport around the ridge.



Bisgard Fig. 5.1 (p. 281): a path γ and downhill deformations γ_1, γ_2.

Quoted caption (Bisgard, Fig. 5.1): "...steep terrain... deformed more... flat terrain."

VMS closure intuition: any procedure that monotonically reduces "cost" will tend to collapse whole families of optional routes into a smaller set of survivors. The "high point" that refuses to go away under legal deformation is the structural bottleneck.

3) Flat directions create long-lived transition behavior

A key visual: when contour lines spread out, the landscape is locally flat. Flatness means the gradient is small — so drift can persist without a sharp push toward a new basin.

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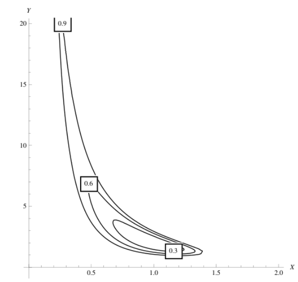


Fig. 2.1 *Shorelines at different levels ($\triangle h = .3$); note how they spread out along the positive y-axis!*

get farther and farther apart. (See Figure 2.1.) Next, recall that $\nabla F(x, y)$ tells us two geometric things about the graph of F at the point (x, y):

(i) The direction of $\nabla F(x, y)$ tells us the direction of maximum increase of F

Bisgard Fig. 2.1 (p. 278): shorelines spreading apart.

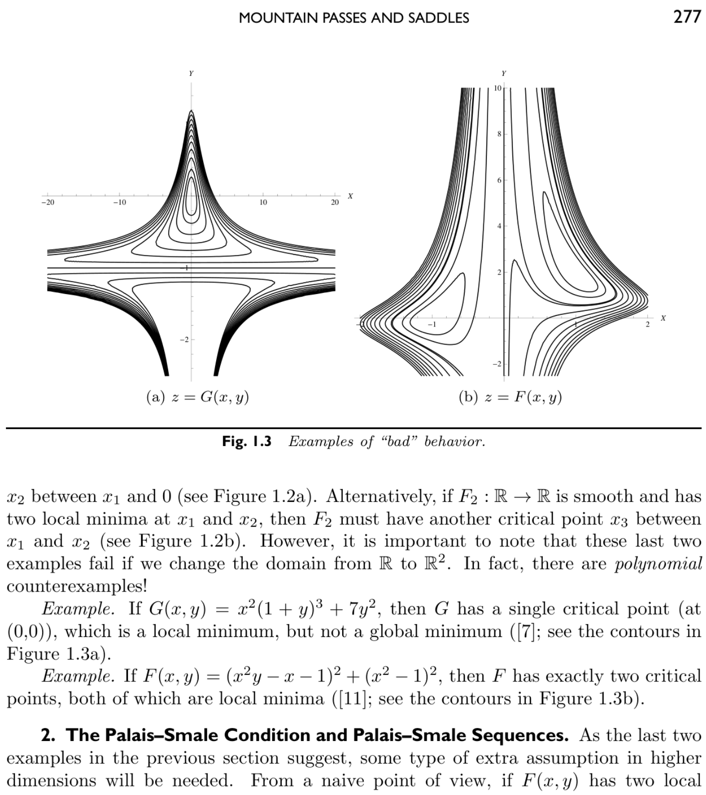
Quoted text (Bisgard, p. 278): “If the contours... are far apart... the landscape is flat... ‖∇F(x,y)‖ must be small.”

VMS closure intuition: this is the simplest way to explain “sticky” transition regions: geometry can create weak-gradient corridors where a state persists, meanders, or broadens before a clean closure outcome appears.

§3 (flat directions) → "This is what's happening in metastable states, broad resonances, and the muon's escape-gap structure (capsule, line 2867 of the loader)."

4) Why naive intuition fails in high dimension

In more than one dimension, you can have multiple basins without a clean, obvious pass between them. This is why closure must include an explicit invariant (something you can't erase by local tinkering).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(a) $z = G(x, y)$ (b) $z = F(x, y)$

Fig. 1.3 *Examples of "bad" behavior.*

x_2 between x_1 and 0 (see Figure 1.2a). Alternatively, if $F_2 : \mathbb{R} \to \mathbb{R}$ is smooth and has two local minima at x_1 and x_2, then F_2 must have another critical point x_3 between x_1 and x_2 (see Figure 1.2b). However, it is important to note that these last two examples fail if we change the domain from $\mathbb{R}$ to $\mathbb{R}^2$. In fact, there are *polynomial* counterexamples!

Example. If $G(x, y) = x^2(1 + y)^3 + 7y^2$, then G has a single critical point (at (0,0)), which is a local minimum, but not a global minimum ([7]; see the contours in Figure 1.3a).

Example. If $F(x, y) = (x^2y - x - 1)^2 + (x^2 - 1)^2$, then F has exactly two critical points, both of which are local minima ([11]; see the contours in Figure 1.3b).

2. The Palais–Smale Condition and Palais–Smale Sequences. As the last two examples in the previous section suggest, some type of extra assumption in higher dimensions will be needed. From a naive point of view, if $F(x, y)$ has two local

Bisgard Fig. 1.3 (p. 277): examples of "bad" behavior (counterintuitive contour geometry).

VMS closure intuition: when your parameter space is high-dimensional, you should expect weird topologies. So "it feels like it should close" is not a method. You need a closure invariant.

In one dimension, two local minima force a maximum somewhere between them — that's just the intermediate value theorem applied to F′. Most people carry that intuition into higher dimensions and expect "two basins ⇒ a saddle between them." In $\mathbb{R}^2$ and beyond, that's false. Bisgard exhibits polynomial counterexamples explicitly: a single critical point that is a local but not a global minimum, given by $G(x,y) = x^2(1+y)^3 + 7y^2$; and a polynomial with exactly two critical points, both of which are local minima — no saddle between them — given by $F(x,y) = (x^2y - x - 1)^2 + (x^2 - 1)^2$. The naive "of course it closes" reflex breaks the moment you leave one dimension.

[FIGURE: Bisgard Fig. 1.3 (p. 277): contour plots of the two polynomial counterexamples.]

Bisgard Fig. 1.3 (p. 277): examples of "bad" behavior (counterintuitive contour geometry).

Quoted text (Bisgard, p. 277): "…these last two examples fail if we change the domain from $\mathbb{R}$ to $\mathbb{R}^2$. In fact, there are polynomial counterexamples!"

VMS closure intuition: in high-dimensional route space, your gut-check "obviously this should close" is not a method. Two stable configurations do not automatically have a saddle gate between them, and a single basin does not automatically mean a global minimum. Closure has to be named — an explicit invariant that survives every legal refinement step — not inferred from the topology of the basins. This is why VMS doesn't say "the system finds a closed configuration"; it names the invariant (e.g., the integer winding number $n \in \mathbb{Z}$ on a closed loop, or the action gap $\Delta S/S_0$ at an escape saddle) and the refinement moves that preserve it.

Where this shows up in VMS (optional cross-reference): admissibility (Particle Mechanics §1) names the integer phase condition as a stated invariant rather than arguing from existence; the muon capsule treats the prefactor Q as a quantity that has to be derived from loop geometry rather than assumed from "the system relaxes." Both moves are answers to the high-dimensional pitfall this section illustrates.

5) Intersection is the invariant (closure in one picture)

Bisgard’s saddle-point section gives a single graphic that explains an invariant intersection under downhill-only deformation. If a deformation can only decrease F, some intersections cannot be removed.

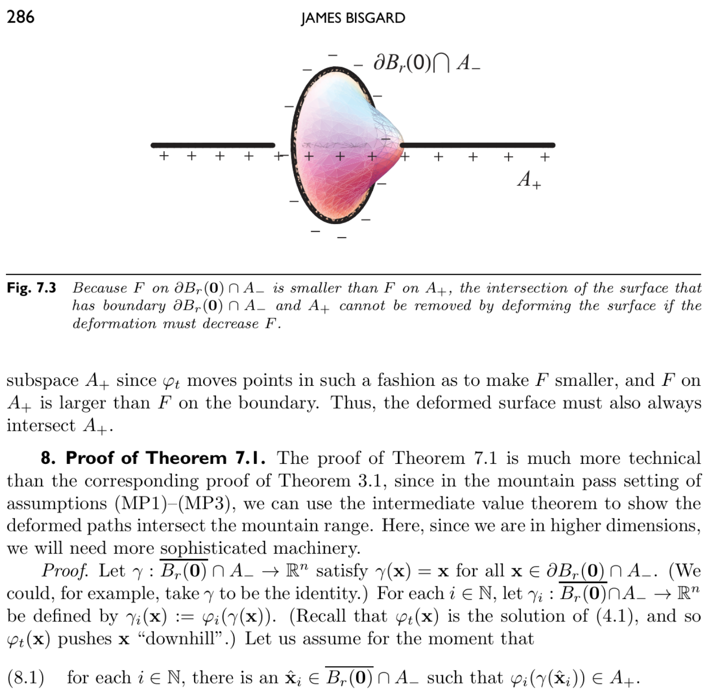

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Fig. 7.3 *Because F on $\partial B_r(\mathbf{0}) \cap A_-$ is smaller than F on A_+, the intersection of the surface that has boundary $\partial B_r(\mathbf{0}) \cap A_-$ and A_+ cannot be removed by deforming the surface if the deformation must decrease F.*

subspace A_+ since φ_t moves points in such a fashion as to make F smaller, and F on A_+ is larger than F on the boundary. Thus, the deformed surface must also always intersect A_+.

8. Proof of Theorem 7.1. The proof of Theorem 7.1 is much more technical than the corresponding proof of Theorem 3.1, since in the mountain pass setting of assumptions (MP1)–(MP3), we can use the intermediate value theorem to show the deformed paths intersect the mountain range. Here, since we are in higher dimensions, we will need more sophisticated machinery.

Proof. Let $\gamma : \overline{B_r(\mathbf{0})} \cap A_- \to \mathbb{R}^n$ satisfy $\gamma(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial \overline{B_r(\mathbf{0})} \cap A_-$. (We could, for example, take γ to be the identity.) For each $i \in \mathbb{N}$, let $\gamma_i : \overline{B_r(\mathbf{0})} \cap A_- \to \mathbb{R}^n$ be defined by $\gamma_i(\mathbf{x}) := \varphi_i(\gamma(\mathbf{x}))$. (Recall that $\varphi_t(\mathbf{x})$ is the solution of (4.1), and so $\varphi_t(\mathbf{x})$ pushes $\mathbf{x}$ “downhill”.) Let us assume for the moment that

(8.1) for each $i \in \mathbb{N}$, there is an $\hat{\mathbf{x}}_i \in \overline{B_r(\mathbf{0})} \cap A_-$ such that $\varphi_i(\gamma(\hat{\mathbf{x}}_i)) \in A_+$.

Bisgard Fig. 7.3 (p. 286): forced intersection under allowed deformation.

Quoted caption (Bisgard, Fig. 7.3): “...intersection... cannot be removed... if the deformation must decrease F.”

VMS closure intuition: define what moves are legal (refinement steps that reduce your closure score), then prove the target structure cannot be erased under those moves. That target structure is what you’re actually “closing.”

We are using the mathematics of what happens when space curvatures and gravitational waves interact. Mathematics of folding-and-focusing geometry, of how smooth paths and waves concentrate into structured patterns, that has been established and accepted for decades.

§5 (intersection invariant) → "This is the template behind 'closure is admissible' — the integer winding number that survives every legal refinement step.”

Putting it together. When VMS says a structure is 'closed,' it means: there's a candidate space of routes, a closure score we want to minimize, refinement steps that legally only reduce that score, and a topological invariant that survives every legal refinement. That invariant is what the framework actually claims about reality. The atlas equations (F0001–F0031) are the specific invariants the framework picks out; this primer is just the visual math language those invariants are written in.

VMS Dictionary Page (mapping)

This page translates the *visual math language* used in Bisgard into a closure vocabulary suitable for VMS communication. It does not add new VMS derivations; it only maps the pictures to the words.

Landscape height F: Your closure score / action / inconsistency measure. It's the thing you try to reduce during refinement.

Contour lines: Equal-score sets. Where contours are close: strong gradients (fast tightening). Where contours spread: weak gradients (slow drift / transition persistence).

Path γ: One candidate route (one candidate history/trajectory/solution family) before refinement.

Optional paths: Many different γ exist between two regions. Optionality is not 'freedom' — it's a set of candidates that closure must filter.

Downhill deformation φ_t: Any legal refinement step that monotonically reduces the closure score. (Bisgard models this as a bounded negative-gradient flow.)

High point along a deformed path: The unavoidable bottleneck. The location where even the best-refined route must still pay the 'ridge' cost.

Mountain pass: A forced bottleneck between basins. In closure language: a necessary transition state / gate.

Saddle geometry + intersection: The template for a closure invariant: some intersections cannot be erased if every refinement step must reduce score.

PS-sequence / nonconvergence (intuition): A warning sign: sequences can look like they're 'approaching closure' while running off to infinity in parameter space if your setup lacks compactness/constraints.

Citation note: all figures and quoted snippets above are from Bisgard (2015) as cited in captions and quotes. See the provided PDF for full context and proofs.