Mechanics — Math Walk-Through

Purpose. Derive the working kernels for bending, travel-time delay, and clock drift in the VMS framework, and show the smooth-limit bridge to the standard weak-field results.

Conventions and Locks

Clarifications & Conventions (alignment with Narrative and Appendix)

• "Route action" is the display-area action $S[path] \equiv \int A\_d \, ds$. We compare routes by S; phase appears only when needed via $\varphi\_phase = S/S_0$.

• Background/cost map n(x): think of n as a smooth cost map tied to missing space; higher n means "this step hides a bit more display-area than average." It is not a new substance.

• Link to potential and sign convention: set $n(x) = 1 + \eta_0 \varphi(x)$ with $\varphi = \Phi/c^2$ and $\Phi < 0$ near mass. With $|\eta_0|$ fixed by deflection ($|\eta_0| = 2$), bending is toward the mass and travel-time shifts are positive.

• Scope note: weak + strong field live here. For steep profiles n(x) evaluate the same kernels piecewise; avoid turning points with refinement.

• Single dimensional scale: $S_0 = \hbar$ (SI-locked).

• Projected display-area: $A\_d = \pi r^2$ (orthographic; no 4π shells).

• Route action: $S[path] = \int A\_d \, ds$ (ratios first; phase = S/S_0 when needed).

• Speed of advance: c; Newton's constant: G.

Background Model and Ray Equation

A stationary mass is treated as a closed circulation whose far-field effect is summarized by a smooth cost map n(x). Adopt the WKB/eikonal ansatz $U = A e^{i k_0 S}$. Collecting powers of k_0 yields the eikonal and transport equations:

$$|\nabla S|^2 = n(x)^2$$

$$2 \nabla S \cdot \nabla A + A \nabla^2 S = 0$$

Rays follow the normal to S. Fermat form $\delta\int n \, ds = 0$ gives the ray equation:

$$\frac{d}{ds}(n \, t) = \nabla n \quad \text{(t is the unit tangent)}$$

1. Bending Kernel

Take the component perpendicular to the ray:

$$d\theta/ds = (1/n)\ (\nabla_\perp n) \approx \nabla_\perp \ln n \qquad \text{(small angles)}$$

Integrate along the unperturbed straight line to get the net deflection vector:

$$\vec{\alpha} \approx \int (\nabla_\perp \ln n)\ ds$$

Weak-field model: $n(x) = 1 + \eta_0\ \varphi(x)$, with $\varphi = \Phi/c^2$ and $\Phi = -GM/r$. For $|\eta_0\ \varphi| \ll 1$, $\ln n \approx \eta_0\ \varphi$, so:

$$\vec{\alpha} \approx \eta_0 \int \nabla_\perp\ \varphi(x)\ ds$$

For a grazing pass with impact parameter b:

$$|\alpha| \approx 2\ |\eta_0|\ GM\ /\ (b\ c^2)$$

Matching the standard weak-field value $|\alpha| = 4\ GM/(b\ c^2)$ fixes $|\eta_0| = 2$ (sign chosen to match deflection toward mass).

1A. Strong-Field: Photon-Sphere and Capture (dictionary check)

For steep, spherically symmetric profiles n(r) that grow sharply as r decreases, closed circular null routes appear when transverse curvature balances forward advance. Define the critical impact parameter b_c as the smallest value for which a passing route still escapes; as $b \to b\_c^+$, the deflection $|\alpha|$ diverges and multiple "looped" images arise. For b < b_c, routes are captured.

In the GR-limit dictionary this reproduces the familiar values (photon sphere and critical impact parameter):

$$b\_c = 3\sqrt{3}\ GM\ /\ c^2$$

Evaluation uses the same kernel $\vec{\alpha} = \int \nabla_\perp \ln n\ ds$ on the steep n(r) profile; the divergence near b_c is a geometric property of the logarithmic integral in a rapidly varying cost map.

$$r\_ph = 3\ GM\ /\ c^2$$

2. Travel-Time Delay

Propagation time increment relative to empty space:

$$\Delta t = (1/c) \int [\ n(x) - 1\ ]\ ds \approx (\eta_0/c) \int \varphi(x)\ ds$$

For a point mass on a straight-line path with closest approach b, evaluation gives the standard logarithmic form after using $|\eta_0| = 2$:

$$\Delta t \approx (2\ GM\ /\ c^3)\ \ln(\ (r\_S + r\_R + D)\ /\ (r\_S + r\_R - D)\ )$$

2A. Observer Dictionary — Lensing Map and Time-Delay Decomposition

For far-field geometry (thin-lens approximation) with distances D_l (observer→lens), D_s (observer→source), D_{ls} (lens→source), connect the deflection kernel to observed angles:

Here $\alpha(\theta) \approx (1/D_l) \int \nabla_\perp \ln n \, ds$ written along the unperturbed path. Relative arrival time between two images i and j splits into geometric and potential parts:

The projected potential ψ is linked to ln n via the same delay kernel $\Delta t = (1/c) \int (n - 1)\, ds$. This dictionary is only a re-expression of the kernels above for inference use; no new parameters are introduced.

$$\Delta t_{ij} = (D_\Delta / c) \left[\tfrac{1}{2}(\theta_i - \beta)^2 - \tfrac{1}{2}(\theta_j - \beta)^2 - \psi(\theta_i) + \psi(\theta_j) \right]$$

$$\beta = \theta - (D_{ls}/D_s)\, \alpha(\theta)$$

r_S, r_R: source/receiver distances from the mass; D: separation.

3. Clock Drift (Redshift / Time-Dilation)

In a stationary background the local tick rate obeys, to leading order:

$$d\tau \approx dt \cdot \sqrt{(1 + 2\varphi)} \quad \Rightarrow \quad \Delta\nu/\nu \approx -\Delta\varphi$$

We treat this as the smooth-limit dictionary of the same cost picture (no new parameters).

4. Bound Orbits: Periapsis Precession (Sketch)

For slowly moving matter routes in a central profile $n(r) = 1 + \eta_0\, \varphi(r)$, transverse curvature from $\nabla_\perp \ln n$ produces a small advance of periapsis per orbit. In the weak-field, low-speed limit the result reduces to:

$$\Delta\varpi \approx 6 \pi GM / (a (1 - e^2) c^2) \quad \text{per orbit}$$

Full derivation is provided via an effective potential constructed from the ray equation (Appendix).

5. Rotators and Frame-Dragging (Preview)

With angular momentum J, the background acquires a small odd-parity correction (azimuthal bias). In the smooth-limit dictionary this corresponds to the Lense–Thirring rate:

$$\Omega_{LT} = 2 G J / (c^2 r^3) \quad \text{(axis-aligned, far field)}$$

We treat these corrections as extensions; in bench contexts they are reported as bounds unless a positive detection is required.

6. Validity and Limits

• Eikonal/WKB: $|\nabla n|/n \ll k_0$. In strong field, integrate piecewise and avoid turning points.

• Paraxial expressions are used for closed-form deflection; otherwise evaluate $\vec{\alpha} = \int \nabla_\perp \ln n \, ds$ numerically on n(x).

• No new dimensional scales are introduced beyond $S_0 = \hbar$; c and G are acceptance locks.

7. Replication Examples

Solar grazing deflection: with b ≈ R_⊙ and M_⊙,

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|α| ≈ 4 G M_⊙ / ( R_⊙ c² ) ≈ 1.75 arcsec
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Shapiro delay at superior conjunction: logarithmic microsecond-scale shift with the same η_0.

Near-Earth gravitational redshift: $\Delta\nu/\nu \approx g\,h / c^2$ at laboratory heights (Pound–Rebka scale).

Appendix Pointers

• Eikonal working: gradients of U = A e^{i k_0 S}; transport; derivation of d/ds(n t) = ∇n.

• Deflection integral along straight-line parameterization (x=b, z=s).

• Delay integral and endpoint geometry for the logarithm.

• Time-dilation mapping to $\sqrt{(1 + 2\varphi)}$.

• Periapsis advance via effective potential.

• Rotators: odd-parity term and far-field limit.

Bottom Line

One ontology, one scale ($S_0 = \hbar$), one cost map n(x). The once-set factor that matches solar deflection also fixes delay and clock-shift. No extra knobs appear mid-derivation; kernels are short and SI-locked.

Appendix Setup (prep for detailed Math Appendix)

The companion Appendix carries the line-by-line evaluations alluded to above, using the same symbols and sign conventions stated here. Pointers:

• Rotators: odd-parity correction (far-field Lense–Thirring limit) and bench-style bounds.
• Periapsis precession: effective potential constructed from d/ds(n t) = ∇n and small-eccentricity expansion.
• Time-dilation: local cost-per-time ratio leading to $\sqrt{(1 + 2\varphi)}$ and $\Delta\nu/\nu \approx -\Delta\varphi$.
• Delay kernel: endpoint geometry for the standard log expression; relation to ψ in the thin-lens dictionary.
• Deflection integral in a 1/r potential: straight-line parameterization (x = b, z = s) and extraction of the logarithmic term.