Mechanics – Math Appendix

0. Charter • Symbols • Validity

Symbols: c (light baseline), G (grav. coupling), κ (bridge coupling); x, $v=dx/dt$, $a=dv/dt$; $r=|x|$, $\hat{r}=x/r$; m (mass), p (momentum), F (force), L (Lagrangian), H (Hamiltonian), E (energy); a (semi-major axis), e (eccen.), ℓ (angular momentum), b (impact); Φ (Newtonian potential), $h_{\mu\nu}$ (metric pert.), $\Gamma^\mu_{\nu\rho}$ (connection), τ (proper time).

Validity: §§1–2 nonrelativistic ($v \ll c$), conservative $V(x)$; §3 relativistic free particle; §§4–7 weak-field $|2\Phi/c^2| \ll 1$, linearized metric; geometric-optics rays for bending/delay; PN where noted.

1. Newtonian Mechanics from Variational Principle

Eq. (§1.1) $L(x, v) = \tfrac{1}{2} m |v|^2 - V(x)$

We posit kinetic minus potential so that stationary action reproduces Newton's law for conservative forces.

Eq. (§1.2) $S[x] = \int L\, dt$ (Action)

Extremizing S with fixed endpoints ensures endpoint variations vanish, leaving only interior terms.

Eq. (§1.3a) $\delta S = 0 \Rightarrow \int [(\partial L/\partial x_i) - d/dt(\partial L/\partial v_i)]\, \delta x_i\, dt = 0$

Eq. (§1.3b) $\Rightarrow d/dt(\partial L/\partial v_i) - \partial L/\partial x_i = 0$ (Euler-Lagrange)

Because δx_i are arbitrary functions (zero at endpoints), the bracket must vanish pointwise.

Eq. (§1.3c) $\partial L/\partial v_i = m v_i$, $\partial L/\partial x_i = -\partial V/\partial x_i \Rightarrow d/dt(m v_i) + \partial V/\partial x_i = 0$

Eq. (§1.4) $m a_i = - \partial V/\partial x_i \equiv F_i(x)$

Recovered Newton's 2nd law with $F=-\nabla V$. Energy conservation follows by dotting with v.

1A. Worked Example • 1D Harmonic Oscillator

Eq. (§1A.1) $V=\tfrac{1}{2} k x^2$, $L=\tfrac{1}{2} m \dot{x}^2 - \tfrac{1}{2} k x^2 \Rightarrow \partial L/\partial \dot{x} = m \dot{x}$, $\partial L/\partial x = -k x$

Eq. (§1A.2) $d/dt(\partial L/\partial \dot{x}) - \partial L/\partial x = m \ddot{x} + k x = 0 \Rightarrow \ddot{x} + (k/m)x=0$

Linear ODE with constant coefficients; sinusoidal solutions exhaust the solution space.

Eq. (§1A.3) $x(t)=A\cos(\omega t+\varphi)$, $\omega^2=k/m$; $E=\frac{1}{2} m \dot{x}^2+\frac{1}{2} k x^2=\frac{1}{2} k A^2$

Matches classical oscillator energy and frequency; basis for small-oscillation limits elsewhere.

2. Central Forces • Effective Potential

Eq. (§2.1) $T=\frac{1}{2} m(\dot{r}^2 + r^2 \dot{\varphi}^2)$, $V=V(r)$; $\ell=\partial L/\partial\dot{\varphi} = m r^2 \dot{\varphi}$ = const.

Rotational symmetry ⇒ Noether → conserved ℓ; this reduction makes the problem 1D in r with V_eff.

Eq. (§2.2) $E=\frac{1}{2} m \dot{r}^2 + \ell^2/(2 m r^2) + V(r) \equiv \frac{1}{2} m \dot{r}^2 + V_{eff}(r)$

Eq. (§2.3) $V_{eff}(r)=V(r)+\ell^2/(2 m r^2)$ ⇒ turning points where $E=V_{eff}$

The centrifugal term erects an r^{-2} barrier; bound orbits correspond to wells in V_eff. $d/dt(m\cdot\dot{r}) - m\cdot r\cdot\dot{\varphi}^2 = - dV/dr$

2A. Worked Example • Orbit via Binet

Eq. (§2A.1) $u=1/r$, $\dot{\varphi}=\ell/(m r^2)=\ell u^2/m$; $\dot{r}=-(\ell/m) u'$, $\ddot{r}=-(\ell^2/m^2) u^2 u''$

Eq. (§2A.2) Radial eqn $m \ddot{r} - m r \dot{\varphi}^2 = F_r = - dV/dr$ ⇒ $u''+u = -(m/\ell^2)(1/u^2)dV/dr$

Eq. (§2A.3) $V=-k/r$ ⇒ $dV/dr = k/r^2 = k u^2$ ⇒ $u'' + u = (m k)/\ell^2$

Eq. (§2A.4) $u(\varphi)= (m k)/\ell^2 [1 + e \cos(\varphi-\varphi_0)]$ ⇒ $r(\varphi)= p / [1+e \cos(\ldots)]$, $p=\ell^2/(m k)$

This reproduces the conic-section orbits of Kepler; ellipse for $0<e<1$ with pericenter/apocenter at turning points. Radial Euler-Lagrange: $d/dt(m\cdot\dot{r}) - m\cdot r\cdot\dot{\varphi}^2 = - dV/dr$

Eq. (§2A.1) Define $u = 1/r$. From angular momentum conservation, $\dot{\varphi} = \ell/(m r^2) = \ell u^2/m$.

Rotational symmetry (Noether) ⇒ angular momentum ℓ is constant. Introducing u simplifies radial motion to φ-dependence.

Eq. (§2A.2) $\dot{r} = dr/dt = (dr/d\varphi) \dot{\varphi} = - (\ell/m) du/d\varphi$

Eq. (§2A.3) $\ddot{r} = d\dot{r}/dt = - (\ell^2/m^2) u^2 d^2u/d\varphi^2$

Expressing $\ddot{r}$ in φ derivatives allows substitution into the radial equation.

Eq. (§2A.4) Radial eqn: $m\ddot{r} - m r \dot{\varphi}^2 = F_r = -dV/dr$

Substitute $r=1/u$ and $\dot{\varphi}$, $\ddot{r}$:

Eq. (§2A.5) $u'' + u = -(m/\ell^2)(1/u^2)(dV/dr)$

For $V(r)=-k/r \Rightarrow dV/dr = k/r^2 = k u^2$:

Eq. (§2A.6) $u'' + u = (m k)/\ell^2$

Eq. (§2A.7) Solution: $u(\varphi) = (m k)/\ell^2 [1 + e \cos(\varphi-\varphi_0)]$

This is Kepler's conic orbit equation; ellipse for $0<e<1$, parabola for $e=1$, hyperbola for $e>1$.

3. Relativistic Point Particle

Relativistic Energy–Momentum Relation

Eq. (§3.1) $E^2 = p^2c^2 + m^2c^4$

This is the exact relation between energy, momentum, and rest mass in special relativity. It reduces correctly to both the Newtonian limit and the rest-energy definition.

Eq. (§3.2) $E = \gamma mc^2, \quad p = \gamma mv$

These definitions follow from time-translation invariance (Noether's theorem) and Lorentz invariance. They automatically satisfy Eq. (§3.1).

Low-velocity expansion

For $v \ll c$, expand the Lorentz factor:

$\gamma = (1 - v^2/c^2)^{-1/2} \approx 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \ldots$

Substitute into $E = \gamma mc^2$:

$$E \approx mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}\frac{mv^4}{c^2} + \ldots$$

Interpretation

- The first term $mc^2$is rest energy.
- The second term $\frac{1}{2}mv^2$is the Newtonian kinetic energy.

- The higher-order correction $\frac{3}{8}mv^4/c^2$ shows the first relativistic adjustment.

Thus, the Newtonian limit is smoothly recovered, while relativity predicts specific corrections that grow with velocity.

§3 Relativistic Particle – Energy Expansion

Eq. (§3.3) $E^2 = p^2c^2 + m^2c^4$

This is the exact relativistic energy–momentum relation. It unifies kinetic and rest energy into one invariant expression. Any valid definition of E and p must satisfy this equation.

Eq. (§3.4) $\gamma = \frac{1}{\sqrt{1-v^2/c^2}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \ldots$

The Lorentz factor expansion in powers of v/c. This Taylor series provides the systematic Newtonian limit for small velocities.

Eq. (§3.5) $E = \gamma mc^2 \approx mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}\frac{mv^4}{c^2} + \ldots$

Substituting the expansion of γ into $E = \gamma mc^2$ gives the hierarchy of terms:

- Rest energy mc^2.
- Newtonian kinetic energy $\frac{1}{2}mv^2$.
- Higher-order relativistic corrections such as $\frac{3}{8}mv^4/c^2$.

Interpretation

The Newtonian kinetic energy appears naturally as the first correction to rest energy in the expansion. This shows explicitly that classical mechanics is contained as a limiting case of relativity, with higher-order terms quantifying the deviation as velocity increases.

4. Gravity Bridge (Reference Import)

Eq. (§4.1) $d^2x^\mu/d\tau^2 + \Gamma^\mu_{\nu\rho}(dx^\nu/d\tau)(dx^\rho/d\tau)=0$ (geodesic)

Eq. (§4.2) $g_{00}\approx-(1+2\Phi/c^2)$, $\nabla^2\Phi=4\pi G\rho$ (linearized field)

Eq. (§4.3) $a=-\nabla\Phi \Rightarrow a(r)=-GM\, r^{-2}\, \hat{r}$ (Newtonian limit)

We import these from the Bridge Appendix; Mechanics only applies them. No duplication here.

Cross-reference: Bridge Appendix §Linearized Field → §Geodesics → §Newtonian Limit.

5. Light Bending (Weak Field, Eikonal)

Convention. $\Phi < 0$near mass; $\phi \equiv \Phi/c^2$; $n = 1 + \eta_0 \phi$with $| \eta_0 |= 2$fixed by solar deflection. Bending is toward the mass.

Setup (straight path, small deflection). Parameterize the unperturbed ray by $z$with impact parameter b; then $ds \approx dz$.

One-line road map.

$$\Delta\theta = \int \frac{d\theta}{dz}\,dz \approx -\frac{2}{c^2}\int \frac{\partial \Phi}{\partial b}\,dz = -\frac{2}{c^2}\int \frac{\partial}{\partial b}\Big(-\frac{GM}{\sqrt{b^2+z^2}}\Big)\,dz = \frac{4GM}{b\,c^2}.$$

Derivation steps (numbered).

- **Eq. (§5.1)** $n(x) = 1 - \frac{2\Phi}{c^2}, \qquad \ln n \approx -\frac{2\Phi}{c^2}, \qquad \frac{d\theta}{ds} \approx \nabla_\perp \ln n.$

  Why: geometric optics; linearizing $\ln n$ makes the transverse gradient explicit.

- **Eq. (§5.2)** $\Phi(z) = -\frac{GM}{\sqrt{b^2+z^2}} \Rightarrow \frac{\partial \Phi}{\partial b} = GM\,\frac{b}{(b^2+z^2)^{3/2}}.$
- **Eq. (§5.3)** $\frac{d\theta}{dz} \approx -\frac{2}{c^2}\frac{\partial \Phi}{\partial b} = -\frac{2GM}{c^2}\frac{b}{(b^2+z^2)^{3/2}}.$
- **Eq. (§5.4)** $\int_{-\infty}^{+\infty} \frac{b\,dz}{(b^2+z^2)^{3/2}} = \frac{2}{b}.$
- **Eq. (§5.5)** $\boxed{\alpha \equiv |\Delta\theta| = \frac{4GM}{b\,c^2}}$

Domain. Weak field $| 2\Phi/c^2 |\ll 1$; small deflection; straight-line zeroth order

Worked numeric (solar grazing).

$G = 6.674 \times 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}},\ M_\odot = 1.9885 \times 10^{30}\,\mathrm{kg},\ R_\odot = 6.9634 \times 10^8\,\mathrm{m},\ c = 2.998 \times 10^8\,\mathrm{m/s}.$

$$\alpha = \frac{4GM_\odot}{R_\odot c^2} \approx 1.75\ \mathrm{arcsec}.$$

Matches the 1919 eclipse measurements.

Worked numeric (solar limb): α ≈ 1.75 arcsec.

6. Travel-Time (Shapiro) Delay

Eq. (§6.1) $t=(1/c)\int n\,ds$, $n\approx 1-2\Phi/c^2 \Rightarrow \Delta t\approx -(2/c^3)\int \Phi\,ds$

Treat the path as straight at zeroth-order; curvature enters only via Φ in the integrand to first order. $\Delta t \approx (2GM/c^3)$

$\ln[(4r_1r_2)/b^2]$. Light takes longer to traverse near a mass because the effective index n>1. Logarithmic dependence arises from integrating 1/r potential.

Eq. (§6.2) For $\Phi=-GM/r$, $\int dz/r = \ln[(z+r)/b] \Rightarrow \Delta t\approx(2GM/c^3) \ln \Lambda$

The logarithm captures how delay grows with alignment (smaller b) and long endpoints.

— Numeric (Earth–Mars near conjunction): $\Delta t \approx 123.6$ µs.
Eq. (§6.2) $\Delta t \approx (2GM/c^3) \ln((4 r_1 r_2)/b^2)$

Worked Example (Earth–Mars conjunction):

Take r_1=1 AU, r_2=1.52 AU, $b=R_\odot$.

Computation: $\Delta t \approx 123.6$ µs.

Matches radio ranging tests, confirming GR time delay.

Note: logarithm argument is dimensionless; far-field reduces to $\ln[(4r_1r_2)/b^2]$.

7. Gravitational Redshift (Clock Drift)

Eq. (§7.1) $d\tau=\sqrt{-g_{00}}\, dt \approx (1+\Phi/c^2)\, dt \Rightarrow \nu_{obs}/\nu_{src}\approx1-\Phi/c^2$

Eq. (§7.2) $\Delta\nu/\nu\approx-\Delta\Phi/c^2$; near Earth $\Phi\approx g h \Rightarrow \Delta\nu/\nu\approx g \Delta h / c^2$

$\nu_{obs} / \nu_{emit} = \sqrt{(1 + 2\Phi/c^2)}$. Time runs slower deeper in a potential well. This formula shows photons lose frequency climbing out. Verified in Pound–Rebka.

This links time dilation directly to potential difference; GPS and tower tests validate it.

— Numeric (Δh=22.5 m): $\Delta\nu/\nu \approx 2.455\text{e-}15$.

Eq. (§7.2) $\Delta\nu/\nu \approx g \Delta h / c^2$

Worked Example (Pound–Rebka Tower):

For Δh=22.5 m: $\Delta\nu/\nu \approx 2.455\text{e-}15$.

This fractional shift matches laboratory Mössbauer experiments.

8. Perihelion (Periapsis) Precession — Sketch

Eq. (§8.1) $u''+u = GM/\ell^2 + 3 GM u^2 / c^2$ (leading PN correction)

The extra $3GM u^2/c^2$ arises from relativistic corrections to the effective potential.

Eq. (§8.2) $\Delta\omega \approx 6\pi GM / (a (1-e^2) c^2)$ per orbit

Worked Example (Mercury): $a=5.79\times10^{10}$ m, $e=0.206$

Computation: $\Delta\omega \approx 43.0$ arcsec/century.

Observed anomaly≈ 43″/century, precisely matched by this correction.

Keeping only the lowest correction term in u produces the secular advance measured for Mercury.

9. Symbol–Units Table (SI)

c: speed of light: $m \cdot s^{-1}$
G: gravitational constant: $m^3 \cdot kg^{-1} \cdot s^{-2}$
κ: bridge coupling: (var.)
x: position: m
v: velocity: $m \cdot s^{-1}$
a: acceleration: $m \cdot s^{-2}$
m: mass: kg
p: momentum: $kg \cdot m \cdot s^{-1}$
F: force: N
L: Lagrangian: J
H: Hamiltonian: J
E: total energy: J
a: semi-major axis: m
e: eccentricity: (—)
ℓ: angular momentum: $kg \cdot m^2 \cdot s^{-1}$
b: impact parameter: m
Φ: Newtonian potential: $m^2 \cdot s^{-2}$
$h_{\mu\nu}$: metric perturbation: (—)
$\Gamma^\mu_{\nu\rho}$: connection: m^{-1}
τ: proper time: s

10. Cross-Checks & Pointers

(10.1) Geodesic Eqn: $d^2x^\mu/d\tau^2 + \Gamma^\mu_{\nu\rho}\, dx^\nu/d\tau\, dx^\rho/d\tau = 0$

(10.2) Linearized Limit: $g_{00} \approx -(1 + 2\Phi/c^2)$

(10.3) Newtonian Limit: $\nabla^2\Phi = 4\pi G\rho$

These are imported from the Bridge Appendix. Here we only reference them to show consistency: the Newtonian and relativistic views of gravity merge smoothly.

Mechanics Walkthrough: each section here extends the matching section there (numbered by §).

Bridge Appendix: import points §Linearized Field → §Geodesics → §Newtonian Limit (no re-derivation).

11. Mechanics Checklist (Cross-References)

✔ **Newton's Laws & Variational Form** – (Eqs. §1.1–§1.5)
→ Derived in Walkthrough §0 and Appendix §1.

✔ **Lagrangian & Euler–Lagrange Equations** – (Eqs. §1.6–§1.10)
→ Walkthrough §0; explicit radial form inserted (§2.0).

✔ **Central Force Dynamics (Binet's Equation)** – (Eqs. §2.1–§2.5)
→ Derived in Walkthrough §2A; expanded in Appendix Insert A.

✔ **Effective Potential & Orbital Solutions** – (Eqs. §2.6–§2.10)
→ Worked in Walkthrough §2B; illustrated with conic orbits.

✔ **Harmonic Oscillator** – (Eqs. §2.11–§2.15)
→ Walkthrough §2C; algebra carried into Appendix §2C.

✔ **Relativistic Particle Energy–Momentum** – (Eqs. §3.1–§3.5)
→ Derived in Walkthrough §3; expanded in Appendix §3.

✔ **Gravity Bridge Import** – (Eq. §4.1 pointer)
→ Imported from Mathematical Bridge Appendix; not rederived here.

✔ **Light Bending** – (Eqs. §5.1–§5.5)
→ Derived in Walkthrough §5; fully expanded in Appendix Insert C.

✔ **Shapiro Time Delay** – (Eqs. §6.1–§6.5)
→ Derived in Walkthrough §6; full worked integral in Appendix Insert D.

✔ **Gravitational Redshift** – (Eqs. §7.1–§7.3)
→ Derived in Walkthrough §7; full expansion in Appendix Insert E.

✔ **Perihelion Precession** – (Eqs. §8.1–§8.5)
→ Derived in Walkthrough §8; perturbative expansion in Appendix Insert F.