Electromagnetism Math Walk-Through

We pick up directly from the canonical Foundation Math Walk-Through. We import a single dimensional scale $S_0 = \hbar$ from the electron anchor—no new dimensional scales are introduced here. All field structure descends from the base variational framework; we do not retune base locks.

-Foundation Bridge (2-form F): In the main Math Appendix, display-area flux is encoded as a 2-form F with dF = 0 (Bianchi). With action $S[A] = \frac{1}{2} \int F \wedge *F$, Euler–Lagrange yields d*F = 0.

- The Helmholtz potentials Φ, A here are the local representatives, giving $E = -\nabla\Phi - \partial A/\partial t$ and $B = \nabla\times A$ in the source-free Maxwell limit (see Math Appendix §1–§1.5).

0. Setup and Assumptions

Notation & Objects.

Spacetime coordinates $x = (t, \vec{x})$; potentials $A = (\Phi, \vec{A})$; field 2-form F = dA with components $F_{0i}=E_i$, $F_{ij}=-\varepsilon_{ijk} B_k$.

Units & Locks.

Import a single dimensional scale $S_0 = \hbar$ (electron anchor); all other locks are acceptance targets. No retune of base constants.

Continuity from symmetry.

Gauge invariance $A\to A+\nabla\chi$, $\Phi\to\Phi-\partial t\chi$ implies $\partial\rho/\partial t + \nabla\cdot J = 0$. We use this repeatedly in conservation steps.

Display-area flux is encoded as a 2-form F with dF = 0 (Bianchi). With action $S[A] = \frac{1}{2} \int F \wedge *F$ and minimal coupling $\int J\cdot A\, d^4x$, Euler–Lagrange yields d*F = J. Writing F = dA and splitting A = (Φ, A), we obtain $E = -\nabla\Phi - \partial A/\partial t$ and $B = \nabla\times A$, with continuity $\partial\rho/\partial t + \nabla\cdot J = 0$ (from gauge invariance). (See Math Appendix §1–§1.5.)

Geometry → Potentials → Fields (derivation sketch).

We pick up directly from the canonical Foundation Math Walk-Through. We import a single dimensional scale $S_0 = \hbar$ from the electron anchor—no new dimensional scales are introduced. Base locks are not retuned; all EM structure is inherited via the geometric variational framework.

Imports & Units (explicit).

We begin with the anchor constant of the entire framework:

$S_0 = \hbar$

Geometric operators are carried from the base pillar: curvature descriptor $\mathcal{K}$ and oriented loop density Ω.

Define pillar-local potentials: $\Phi[\mathcal{K},\Omega]$ and $A[\mathcal{K},\Omega]$.

Fields: $E = -\nabla\Phi - \partial A/\partial t$, $B = \nabla\times A$.

Sources are geometric descriptors: (ρ, J) from orientation and topology of loops.

Continuity follows directly: $\partial\rho/\partial t + \nabla\cdot J = 0$.

Limit regime: weak curvature / slow variation → Maxwell equations.

1. Maxwell Set from Geometry

Action and variation (explicit).

Field action $S\_field = \frac{1}{2} \int F \wedge {}^*F$; interaction $S\_int = \int J \cdot A\, d^4x$. Vary A: $\delta S = \int (\delta A) \cdot (d{}^*F - J)\, d^4x \Rightarrow d{}^*F = J$; geometric identity $dF = 0$.

Split into E, B equations.

$\nabla \cdot B = 0$; $\nabla \times E = -\partial B/\partial t$; $\nabla \cdot E = \rho/\varepsilon_0$; $\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \partial E/\partial t$.

Wave equation and dispersion.

Take curl of Faraday, substitute Ampère–Maxwell: $\nabla^2 E - \mu_0 \varepsilon_0 \partial^2 E/\partial t^2 = 0$; plane waves $E = E_0 e^{i(k \cdot x - \omega t)}$ enforce $\omega = c |k|$ with $c = (\mu_0 \varepsilon_0)^{-1/2}$.

Energy–momentum (for later optics power).

Clarification (measurement bridge): We are not importing "energy" or "momentum" as primitives. In this pillar they arise from the same geometric action via Noether; the familiar energy density $u = \frac{1}{2}(\varepsilon_0 E^2 + B^2/\mu_0)$ and Poynting vector $S = (1/\mu_0) E \times B$ are used only as measurement bridges to lab observables (power, intensity, flux). The only dimensional scale admitted is $S_0 = \hbar$ (set once at the electron anchor); vacuum identities ($k\_e = 1/(4\pi \varepsilon_0)$, $c = (\mu_0 \varepsilon_0)^{-1/2}$) are limit locks, not inputs to the dynamics.

$u = \frac{1}{2}(\varepsilon_0 E^2 + B^2/\mu_0)$, $\vec{S} = (1/\mu_0) \vec{E} \times \vec{B}$. These transport relations are used in diffraction and imaging budgets.

The stress–energy from S_field yields the Poynting vector $S = (1/\mu_0) E \times B$ and energy density $u = \frac{1}{2}(\varepsilon_0 E^2 + B^2/\mu_0)$, used later for optical power budgets (Appendix §1.5).

Energy–momentum & Poynting (one line).

Start: $\delta(S\_field + S\_int) = 0$ with $S\_field = \frac{1}{2} \int F \wedge {}^*F$, $S\_int = \int J \cdot A\, d^4x$. Variation δA gives $d{}^*F = J$; closure $dF = 0$ is geometric. In components: $\nabla \cdot B = 0$, $\nabla \times E = -\partial B/\partial t$, $\nabla \cdot E = \rho/\varepsilon_0$, $\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \partial E/\partial t$. Taking $\nabla \times$ of Faraday and using Ampère–Maxwell yields the wave equation $\nabla^2 E - \mu_0 \varepsilon_0 \partial^2 E/\partial t^2 = 0$ with $c = (\mu_0 \varepsilon_0)^{-1/2}$. Plane-wave ansatz $E = E_0 e^{i(k \cdot x - \omega t)}$ enforces $\omega = ck$.

From action to Maxwell (steps).

We recover the Maxwell set under weak curvature / slow variation.

Gauss–B: $\nabla \cdot B = 0$ (vector identity).

Faraday's Law: $\nabla \times E = -\partial B/\partial t$.

Gauss–E: $\nabla \cdot E = \rho/\varepsilon_0$; in statics gives Poisson's equation.

Ampère–Maxwell: $\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0\, \partial E/\partial t$.

Wave equation: $\nabla^2 E - \mu_0 \varepsilon_0\, \partial^2 E/\partial t^2 = 0$, speed $c = (\mu_0 \varepsilon_0)^{-1}/^2$.

Box 1 (Maxwell's Equations, see Math Appendix §1):

$\nabla \cdot E = \rho/\varepsilon_0$

$\nabla \cdot B = 0$

$\nabla \times E = -\partial t B$

$\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0\, \partial t E$

Box 2 (EM Wave Equation, see Math Appendix §1.5):

$\nabla^2 E - (1/c^2)\, \partial^2 E/\partial t^2 = 0$

2. Lorentz Force and Coulomb Limit

Minimal coupling derivation.

$L = \tfrac{1}{2}\, m v^2 + q\, v \cdot A - q\, \Phi \Rightarrow m\, a = q\, (E + v \times B)$. Gauge choices (Coulomb/Lorenz) do not change observables.

Poisson → Coulomb.

Static: $\nabla \cdot E = \rho/\varepsilon_0$, $E = -\nabla \Phi \Rightarrow \nabla^2 \Phi = -\rho/\varepsilon_0$. For $\rho = q\, \delta^3(x)$: $\Phi = k\_e\, q/r$, $E = k\_e\, q\, \hat{r} / r^2$; $k\_e = 1/(4\pi\, \varepsilon_0)$.

Static: $\nabla \cdot E = \rho/\varepsilon_0$ and $E = -\nabla \Phi \Rightarrow \nabla^2 \Phi = -\rho/\varepsilon_0$. Point charge $\rho = q\, \delta(x) \Rightarrow \Phi = k\_e\, q/r$ and $E = k\_e\, q\, \hat{r} / r^2$ with $k\_e = 1/(4\pi\, \varepsilon_0)$ (Appendix §2.2).

Coulomb from Poisson.

With $L = \tfrac{1}{2}\, m v^2 + q\, v \cdot A - q\Phi$, Euler–Lagrange gives $m\, a = q(E + v \times B)$. Gauge invariance $A \to A + \nabla \chi$, $\Phi \to \Phi - \partial t \chi$ leaves equations invariant (Appendix §2).

Minimal coupling → Lorentz force.

Start from minimal-coupling Lagrangian: $L = \tfrac{1}{2} m v^2 + q\, v \cdot A - q\Phi$.

Gauge invariance is preserved under transformations $\Phi \to \Phi - \partial t \chi$, $A \to A + \nabla \chi$.

Euler–Lagrange equations yield: $F = q(E + v \times B)$.

Box 3 (Lorentz Force, see Math Appendix §2):

$F = q(E + v \times B)$

Static point source solution of Poisson's equation gives Coulomb's law.

Box 4 (Coulomb's Law, see Math Appendix §2.2):

$F = k_e\, q_1\, q_2 / r^2$

3. Propagation in Media & Rays

Constitutive relations.
Linear, isotropic media: $D = \varepsilon E$, $B = \mu H$; refractive index $n^2 = \varepsilon_r \mu_r$. In vacuum, $\varepsilon_r = \mu_r = 1$.

Eikonal and ray equation.
Ansatz $E(x) = A(x) e^{\{i S(x)/\lambda\}}$. Leading order: $|\nabla S|^2 = n^2$ (eikonal). Rays follow d/ds $(n \hat{r}) = \nabla n_\perp$; curvature balance $1/a = ||\nabla_\perp \ln n||$.

Interfaces (Snell, Fresnel, TIR, Brewster).
Snell: $n_1 \sin\theta_1 = n_2 \sin\theta_2$. Fresnel: $r\_s = (n_1 \cos\theta_1 - n_2 \cos\theta_2)/(n_1 \cos\theta_1 + n_2 \cos\theta_2)$, $r\_p = (n_2 \cos\theta_1 - n_1 \cos\theta_2)/(n_2 \cos\theta_1 + n_1 \cos\theta_2)$. Brewster: $\tan\theta\_B = n_2/n_1$ (for p-pol). TIR: $\sin\theta\_c = n_2/n_1$ $(n_1 > n_2)$.

Wavelength convention.
Spectroscopy uses vacuum wavelengths unless stated; air values differ by $n(\lambda)$. Balmer anchors follow NIST ASD vacuum lines; reconciled in Calibration.

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Vacuum vs air wavelengths (convention).
Tangential E,H and normal D,B continuity give Fresnel coefficients; Brewster angle $\tan\theta\_B = n_2/n_1$; TIR at $\sin\theta\_c = n_2/n_1$. These feed the polarization section (Appendix §3).

Fresnel/Brewster boundary set.
In linear media: $D = \varepsilon E$, $B = \mu H$ with $n^2 = \varepsilon_r \mu_r$. Eikonal ansatz $E(x) = A(x) e^{\{i S(x)/\lambda\}}$ yields $|\nabla S|^2 = n^2$ and ray equation d/ds $(n \hat{r}) = \nabla n_\perp$. GRIN curvature balance: $1/a = ||\nabla_\perp \ln n||$. Snell's law $n_1 \sin\theta_1 = n_2 \sin\theta_2$ follows from stationary phase at interfaces (Appendix §3).

Constitutive & eikonal chain.
Constitutive relations: $D = \varepsilon E$, $B = \mu H$; refractive index $n^2 = \varepsilon_r \mu_r$.

Eikonal derivation via WKB ansatz gives $|\nabla S|^2 = n^2$.

Snell's Law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$.

Curvature balance: $1/a = ||\nabla_\perp \ln n||$.

Boundary conditions (tangential E,H; normal D,B) $\rightarrow$ Fresnel equations.

Brewster's angle: $\tan\theta_B = n_2/n_1$.

Critical angle for total internal reflection: $\sin\theta_c = n_2/n_1$.

Box 5 (Snell's Law, see Math Appendix §3):
$$n_1\,\sin\theta_1 = n_2\,\sin\theta_2$$

5. Polarization Transport

Representations.

Jones vectors for fully coherent beams; Stokes/Mueller for partial polarization. Stokes $S = (S_0, S_1, S_2, S_3)$; Mueller matrix M gives S_out = M S_in.

Medium effects.

Faraday rotation (small-signal): $\theta = V B L$; birefringence via Δn induces phase retardance $\delta = 2\pi\,\Delta n\,L / \lambda$. Link to ε_eff, μ_eff as geometry-to-medium mapping.

Jones: $\vec{E} = (E\_x, E\_y)$. Stokes $S = (S_0, S_1, S_2, S_3)$ with Mueller matrices M so that S_out = M S_in. Faraday rotation in magnetoactive media: $\theta = V B L$ (small-signal). These models parameterize polarization changes induced by geometry via ε_eff, μ_eff (Appendix §5).

Jones/Stokes/Mueller pipeline.

Jones calculus: field as (Ex,Ey).

Stokes parameters S_0, S_1, S_2, S_3 define polarization state.

Mueller matrices connect lab measurements to polarization state.

Faraday rotation: $\theta = VBL$.

Box 8 (Faraday Rotation, see Math Appendix §5):
$$\theta = V\,B\,L$$

4. Wave Optics

Kirchhoff/Huygens integral.
Scalar diffraction integral reduces to Fresnel (near-field) and Fraunhofer (far-field) regimes by stationary-phase. Replace $\lambda \to \lambda/n$ in media.

Canonical patterns.

Single slit: $I(\theta) = I_0\,(\sin\beta/\beta)^2$, $\beta = (\pi a/\lambda)\sin\theta$. Double slit: fringe spacing $\Delta y = \lambda L / d$. Grating: $d\sin\theta\_m = m\,\lambda$.

Kirchhoff integral yields Fresnel (near-field) and Fraunhofer (far-field) limits. Single-slit: $I(\theta) = I_0\,(\sin\beta/\beta)^2$ with $\beta = (\pi a/\lambda)\sin\theta$. Double-slit: $\Delta y = \lambda L/d$; grating: $d\sin\theta\_m = m\,\lambda$. These carry over to media by replacing $\lambda \to \lambda/n$ (Appendix §4–§4.3).

Huygens–Fresnel and regimes.

Huygens–Fresnel integral $\to$ diffraction patterns.

Single-slit diffraction minima: $a\sin\theta = m\lambda$.

Double-slit interference spacing: $\Delta y = \lambda L/d$.

Diffraction grating: $d\sin\theta m = m\lambda$.

Box 6 (Double-Slit Interference, see Math Appendix §4):
$\Delta y = \lambda L / d$

Box 7 (Diffraction Grating, see Math Appendix §4.3):
$d\sin\theta m = m\lambda$

6. Imaging (Paraxial Optics)

ABCD law.

Ray vector (x, θ) propagates by $[x_2; \theta_2] = [[A,B],[C,D]]\,[x_1; \theta_1]$ with determinant $AD - BC = 1$. Thin lens: $1/f = 1/d\_o + 1/d\_i$; magnification $M = -d\_i/d\_o$. GRIN handled by slice composition.

Paraxial propagation through elements via ABCD matrices; thin-lens equation $1/f = 1/d\_o + 1/d\_i$; magnification $M = -d\_i/d\_o$. Determinant constraint $AD - BC = 1$ preserves optical invariant; extends to GRIN via slice composition (Appendix §6).

ABCD matrices & invariants.

Ray transfer (ABCD) matrices in paraxial optics.

Thin lens equation: $1/f = 1/do + 1/di$.

Magnification: $M = -di/do$.

Box 9 (Thin Lens Equation, see Math Appendix §6):
$1/f = 1/do + 1/di$

7. Spectroscopy Anchors

Rydberg formula.

$1/\lambda = R\_\infty \, (1/n_1^2 - 1/n_2^2)$. Balmer anchors (vacuum; NIST ASD): Hα 656.281 nm, Hβ 486.133 nm, Hγ 434.047 nm.

Ratio-first use.

Use ratios (e.g., Hβ/Hα = 20/27 ideal) when they beat absolute calibration error; carry uncertainties through to acceptance bands.

Hydrogen: $1/\lambda = R\_\infty \, (1/n_1^2 - 1/n_2^2)$. Balmer anchors (vacuum; NIST ASD): Hα 656.281 nm, Hβ 486.133 nm, Hγ 434.047 nm. Prefer ratio-first predictions (e.g., Hβ from Hα via 20/27) when they beat absolute calibration error (Appendix §7).

Rydberg/anchors (vacuum).

Hydrogen spectrum via Rydberg formula: $1/\lambda = R\infty \, (1/n_1^2 - 1/n_2^2)$.

Balmer anchors: Hα, Hβ, Hγ. Ratio-first prediction e.g. Hβ from Hα via 20/27.

Box 10 (Rydberg Formula, see Math Appendix §7):

$$1/\lambda = R\infty \, (1/n_1^2 - 1/n_2^2)$$

8. Corrections (Torsion/Shear)

Parameterization & bounds.

Small, dimensionless budgets enter phase and transport: $\delta\varphi \simeq \oint (\alpha\_T \, T + \alpha\_S \, S)\cdot dl$; $\delta n/n \simeq \beta\_T \, T + \beta\_S \, S$. Bounds traced by closure tolerance J_c; vanish in pure vacuum.

Introduce small, dimensionless budgets for torsion/shear—entering as phase and transport corrections: $\delta\varphi \simeq \oint (\alpha\_T \, T + \alpha\_S \, S)\cdot dl$, $\delta n/n \simeq \beta\_T \, T + \beta\_S \, S$. These are bounded by closure tolerance J_c and are testable; they vanish in pure vacuum (Appendix §8).

Bounded perturbations (how).

Torsion/shear operators define bounded corrections.

Perturbations appear in refractive index, boundary phases, polarization transport.

Bounds provide falsifiers if exceeded.

9. Falsifiers & Lab-Math Hooks

Neutral-rotor fingerprint.

Composite neutral (monopole sum zero) aligned to global caustic plane; spin picks orientation. With $\varepsilon > 0$, half-cycles aligned with $+\hat{n}$ are weighted slightly more than opposite half-cycles, giving $\langle \Delta p \rangle \propto \varepsilon\,\sigma\,\cos\theta$ (push/zero/pull). Spin-flip reverses sign; 90° tilt is a true null. Absence sets a sharp upper bound; presence is new physics vs. point-charge intuition.

A composite neutral (net monopole zero) still aligns to the global caustic plane. Spinning a non-magnetic rotor in that plane gates orientation; the model predicts a tiny cycle-odd response: $\langle \Delta p \rangle \propto \varepsilon\,\sigma\,\cos\theta$ (push/zero/pull). Spin-flip reverses sign; wrong facing yields null. Absence sets a sharp upper bound; presence is a clean falsifier of 'point-charge only' intuition (Appendix §9).

Neutral-rotor (explicit).

Neutral rotor predicts orientation-gated forces.

Push / zero / pull triad arises from asymmetry $\propto \varepsilon\sigma\,\cos\theta$.

Spin-flip reverses sign. Null cones appear as discrete angular zeros.

Spectroscopy ratio bands: calibrated vs ratio-first predictions.

Box 11 (Neutral Rotor Prediction, see Math Appendix §9):

$\langle \Delta p \rangle$ $\varepsilon\,\sigma\,\cos\theta$

10. Appendix Cross-Refs

Exact mapping to Math Appendix.

Maxwell & wave eqn (§1–§1.5); Lorentz & Coulomb (§2–§2.2); Snell/Fresnel/GRIN (§3); Fresnel–Kirchhoff & gratings (§4–§4.3); Faraday (§5); Paraxial (§6); Rydberg/spectroscopy (§7); Torsion/Shear bounds (§8); Neutral-rotor falsifier (§9). Keep numbers synchronized with the main Appendix index.

Lock statement.

Classical vacuum relations appear here only as limit checks/locks; the pillar's dynamics are derived from geometry with no new dimensional scales beyond $S_0 = \hbar$ (set once at the electron anchor; no retune).

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Exact mapping.

Full Fresnel derivation, Kirchhoff integral, Jones↔Stokes conversions, air-index models.

Worked examples: Coulomb potential, Fresnel/Brewster, diffraction, Faraday rotation, GRIN curvature.

Optional: thin-film transfer matrices.

Wavelength convention: spectroscopy uses vacuum wavelengths unless stated; air values differ by n. Numeric anchors (e.g., Balmer) follow the NIST ASD vacuum list and are reconciled in the pillar Calibration doc.

Balmer anchors (vacuum; NIST ASD): Hα 656.281 nm, Hβ 486.133 nm, Hγ 434.047 nm. Ratio-first predictions (e.g., Hβ from Hα via 20/27) are preferred when they beat absolute calibration error.

Cross-References to Foundation & Appendix (exact): Maxwell set & wave equation (Math Appendix §1–§1.5); Lorentz force & Coulomb (§2–§2.2); Snell/Fresnel/GRIN (§3); Fresnel–Kirchhoff & gratings (§4–§4.3); Faraday (polarization) (§5); Paraxial optics (§6); Rydberg/spectroscopy (§7); Torsion/Shear corrections & bounds (§8); Neutral-rotor falsifier (§9). Keep these section numbers synchronized with the main Appendix index.

Classical vacuum relations appear here only as limit checks/locks; the pillar's dynamics are derived from geometry with no new dimensional scales beyond $S_0 = \hbar$ (set once at the electron anchor; no retune).