

## Electromagnetism — Math Appendix (Readable Equations Edition)

Charter. This appendix mirrors the Electromagnetism Math Walk-Through step-for-step. For each step we provide full derivations, symbols and units, validity limits, and a replication example with real numbers. No summaries. We import only  $S_0 = \hbar$  from the Foundation calibration and add no new dimensional scales. Vacuum identities ( $\mu_0, \epsilon_0, c$ ) appear only as acceptance locks, not tunables. Cross-refs point back to the Bridge/Foundation appendix where identities are first established. All derivations are strictly based on primitive axioms, with calibration notes for anchors. Equations are set in plain Unicode math for readability.

### Symbol Registry

$S_0$  —  $\hbar$  (Planck's reduced constant), fixed at the electron anchor (Calibration: Electron).

$A^\mu = (\Phi/c, \vec{A})$  — 4-potential (SI:  $\Phi$  in volts;  $\vec{A}$  in V·s/m).

$F = dA$  — field 2-form; components  $F_{0i} = E_i$ ,  $F_{ij} = -\epsilon_{ijk} B_k$  (covariant blocks use  $c=1$ ; SI factors enter post-variation).

$E, B$  — Electric and magnetic fields (SI: V/m, T).

$J^\mu = (c\rho, \vec{J})$  — 4-current (SI: A/m<sup>2</sup>);  $\partial_\mu J^\mu = 0$ .

$\mu_0, \epsilon_0, c$  — Vacuum locks;  $c = 1/\sqrt{(\mu_0 \epsilon_0)}$  (acceptance check only).

$\eta, \epsilon^{\{\mu\nu\alpha\beta\}}$  — Metric  $(-, +, +, +)$ , Levi-Civita ( $\epsilon^{\{0123\}} = +1$ ).

$\star$  — Hodge star (Bridge Appx: Hodge & Duality).

$E_\kappa, T, S, C$  — Curvature energy; torsion/shear budgets; closure/linking/non-intersection constraints (Foundation).

$n(x), k = 2\pi/\lambda$  — Refractive index; wavenumber.

$N_F = a^2/(\lambda z)$  — Fresnel number.

$J_c$  — Closure tolerance,  $J_c = \Delta S / S_0$ , fractional deviation from perfect loop closure (Foundation: Torsion/Shear Budgets).

### 0. Setup and Assumptions

Cross-ref: Bridge Appx — Field 2-form & Variation; Foundation — Axioms A1–A3.

Loop motion transports oriented display-area elements, defined as the obscured transverse area projected along propagation:  $A_d(\Sigma) = \int_{\Sigma} (\hat{n} \cdot d\Sigma)$ . By refinement invariance and Stokes,  $\Phi(\Sigma) = \int_{\Sigma} F$  with  $F = dA$  and  $dF = 0$  (Bianchi). This ensures conservation of display area under Void propagation.

$$F = dA, \quad dF = 0$$

Observer split of  $dF = 0$  (homogeneous pair):

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$$

(Conventions) Covariant blocks use  $c = 1$  so  $F_{0i} = E_i$  and  $F_{ij} = -\epsilon_{ijk} B_k$ ; SI factors enter only in the post-variation bridge (e.g.,  $\partial_{\mu} F^{\{\mu\nu\}} = \mu_0 J^{\nu}$ ).

Validity: weak curvature/slow variation; torsion/shear corrections neglected ( $|T|, |S| \ll 1$ ).  
Electron-scale precision suggests  $J_c \leq 10^{-9}$ .

Replication (consistency): Rectangular loop (1×1 m) with uniform  $B_z = 1$  T:  $\Phi = B \cdot \text{Area} = 1 \text{ T} \cdot \text{m}^2$ .  
Refinement to 2×2 sub-loops preserves the total flux.

## 1. Maxwell Set from Geometry

**Context & notation.** Geometric field 2-form  $\mathbf{F} = d\mathbf{A}$  encodes oriented display-area transport; the homogeneous equations follow from  $d\mathbf{F} = 0$  (Bianchi). We vary the vacuum action; SI units appear only after variation (measurement bridge).

4-potential and current (SI bridge shown explicitly):

$$A^{\mu} = (\Phi/c, \mathbf{A}), \quad J^{\mu} = (c\rho, \mathbf{J})$$

Vacuum action with minimal coupling:

$$S[A] = \frac{1}{2} \int F \wedge \star F - \int J \cdot A \quad d^4x, \quad \text{with} \quad F = dA$$

Variation (boundary term dropped by compact support / decay at  $\infty$ ):

$$\delta F = d(\delta A)$$

$$\delta S_{\text{field}} = \int d(\delta A) \wedge \star F = \int d(\delta A \wedge \star F) - \int \delta A \wedge d\star F$$

$$\delta S_{\text{int}} = - \int \delta A \wedge J$$

$$\delta S = \int \delta A \wedge (d\star F - J) \Rightarrow d\star F = J, \quad dF = 0$$

Index / SI bridge (measurement only):

$$\partial_\mu F^{\{\mu\nu\}} = \mu_0 J^{\{\nu\}}, \quad \partial_\alpha F_{\{\beta\gamma\}} + \partial_\beta F_{\{\gamma\alpha\}} + \partial_\gamma F_{\{\alpha\beta\}} = 0$$

Observer split (E, B) in SI:

$$\nabla \cdot \mathbf{E} = \rho / \epsilon_0, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \partial \mathbf{E} / \partial t$$

Wave equations in source-free vacuum (curl-curl identities shown in the working):

$$\nabla^2 \mathbf{E} - \mu_0 \epsilon_0 \partial^2 \mathbf{E} / \partial t^2 = 0, \quad \nabla^2 \mathbf{B} - \mu_0 \epsilon_0 \partial^2 \mathbf{B} / \partial t^2 = 0$$

$$c = 1 / \sqrt{(\mu_0 \epsilon_0)}$$

Replication (Coulomb limit; non-relativistic, static):

$$\mathbf{F} = q (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \approx q \mathbf{E} \quad (v \ll c)$$

$$\Phi(r) = q / (4\pi \epsilon_0 r), \quad \mathbf{E} = q / (4\pi \epsilon_0 r^2) \hat{h}_r$$

$$F = k_e \cdot e^2 / r^2, \quad k_e = 1 / (4\pi \epsilon_0)$$

Numerical check (CODATA):  $e = 1.602\,176\,634 \times 10^{-19}$  C (exact),  $\epsilon_0 \approx 8.854\,187\,8128 \times 10^{-12}$  F/m  $\rightarrow k_e \approx 8.987\,551\,7923 \times 10^9$  N·m<sup>2</sup>·C<sup>-2</sup>. For  $r = 1$  nm:  $F \approx 2.31 \times 10^{-10}$  N.

Validity:  $v \ll c$ ; static sources; separation large vs. quantum-correction scales (no QED / radiation reaction).

## 2. Lorentz Force — Relativistic and Non-Relativistic

**Context & notation.** Relativistic (4D) first, then the lab 3-vector form. We use plain notation with bold vectors (E, B, v) and minimal symbols. Covariant blocks take  $c=1$ ; SI enters only after the variation.

**Relativistic Lorentz force (covariant form).**

Four-momentum and four-velocity:  $p^\mu = m u^\mu$ ,  $u^\mu = dx^\mu / d\tau$ . Field tensor:  
 $F_{\{\mu\nu\}} = \partial_\mu A_\nu - \partial_\nu A_\mu$ .

$$dp^\mu / d\tau = q F^{\{\mu\}\}_{\{\nu\}} u^\nu \quad (\text{Eq. 2-1})$$

Orthogonality:  $u_\mu (dp^\mu / d\tau) = 0$  (since  $F_{\{\mu\nu\}} = -F_{\{\nu\mu\}}$ ). Gauge invariance:  $A_\mu \rightarrow A_\mu + \partial_\mu \chi$  leaves F and the force invariant.

### Observer split → 3-vector force and power balance.

With  $\mathbf{p} = \gamma m \mathbf{v}$ ,  $\gamma = 1/\sqrt{1-v^2/c^2}$  and  $dt = \gamma d\tau$ , the spatial and temporal parts give:

$$d\mathbf{p}/dt = q (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (\text{Eq. 2-2})$$

$$d(\gamma m c^2)/dt = q (\mathbf{v} \cdot \mathbf{E}) \quad (\text{Eq. 2-3})$$

Non-relativistic limit ( $v \ll c$ ):  $\mathbf{p} \approx m \mathbf{v}$  and  $d(m \mathbf{v})/dt = q (\mathbf{E} + \mathbf{v} \times \mathbf{B})$ .

### Derivation from the particle Lagrangian (explicit steps).

Start with the standard Lagrangian:

$$L = -m c^2 \sqrt{1-v^2/c^2} + q (\mathbf{v} \cdot \mathbf{A}) - q \Phi \quad (\text{Eq. 2-4})$$

Canonical momentum and Euler-Lagrange:

$$\partial L / \partial \mathbf{v} = \gamma m \mathbf{v} + q \mathbf{A} \equiv \mathbf{p}_{\text{can}} \quad (\text{Eq. 2-5})$$

$$d/dt (\partial L / \partial \mathbf{v}) - \partial L / \partial \mathbf{x} = 0 \quad (\text{Eq. 2-6})$$

Using  $\mathbf{E} = -\nabla\Phi - \partial\mathbf{A}/\partial t$  and  $\mathbf{B} = \nabla \times \mathbf{A}$ , and expanding  $d\mathbf{A}/dt = \partial\mathbf{A}/\partial t + (\mathbf{v} \cdot \nabla)\mathbf{A}$ , cancellations yield the Lorentz force:

$$d(\gamma m \mathbf{v})/dt = q (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (\text{Eq. 2-7})$$

### Replication (cyclotron motion; uniform $\mathbf{B}$ , $\mathbf{E} = 0$ ).

Centripetal balance gives the cyclotron frequency:

$$\omega_c = q B / (\gamma m) \quad (\text{Eq. 2-8})$$

Non-relativistic electron ( $\gamma \approx 1$ ) in  $B = 1 \text{ T}$ :

$$\omega_c \approx (e/m_e) B \approx 1.75882000838 \times 10^{11} \text{ s}^{-1} \quad (\text{CODATA 2022}) \quad (\text{Eq. 2-9})$$

### Validity & limits.

Point charge; classical regime; fields smooth on the formation time. Radiation reaction and quantum corrections neglected unless required by precision.

## 3. Propagation in Media & Rays — Eikonal/WKB and Interfaces

**Setup & scope.** Derive rays from the scalar Helmholtz model with slowly varying index  $n(x)$  via a WKB ansatz; then apply interface boundary conditions to get Fresnel coefficients and special angles. No LaTeX macros; all formulas are plain and readable.

### Helmholtz equation and WKB ansatz.

$$\nabla^2 U + k_0^2 \cdot n(x)^2 \cdot U = 0 \quad (\text{Eq. 3.1})$$

$$U(x) = A(x) \cdot \exp(i \cdot k_0 \cdot S(x)) \quad (\text{Eq. 3.2})$$

Compute gradients and collect orders in  $k_0$ :

$$\nabla U = \exp(i k_0 S) \cdot [ \nabla A + i k_0 \cdot A \cdot \nabla S ] \quad (\text{Eq. 3.3})$$

$$\nabla^2 U = \exp(i k_0 S) \cdot [ \nabla^2 A + 2 i k_0 \cdot (\nabla S \cdot \nabla A) + i k_0 \cdot A \cdot (\nabla^2 S) - k_0^2 \cdot A \cdot |\nabla S|^2 ] \quad (\text{Eq. 3.4})$$

Match powers of  $k_0$  (leading two):

$$|\nabla S|^2 = n^2(x) \quad (\text{Eq. 3.5})$$

$$2 (\nabla S \cdot \nabla A) + A \cdot (\nabla^2 S) = 0 \quad (\text{Eq. 3.6})$$

Interpretation:  $S$  is the eikonal (optical path). The transport equation conserves flux along rays.

**Rays from Fermat's principle.**

$$\delta \int n(x) ds = 0 \quad (\text{Eq. 3.7})$$

Euler-Lagrange gives the ray evolution with unit tangent  $t = dx/ds$ :

$$d/ds [ n \cdot t ] = \nabla n \quad (\text{take the component } \perp \text{ to } t) \quad (\text{Eq. 3.8})$$

$$\text{Ray curvature: } 1/a = || \nabla \perp \ln n || \quad (\text{Eq. 3.9})$$

Here  $a$  is the local radius of curvature;  $\nabla \perp$  is the component of  $\nabla$  perpendicular to the ray.

**Planar interface boundary conditions (no free surface charge/current).**

$$\hat{n} \cdot (D_2 - D_1) = 0, \quad \hat{n} \cdot (B_2 - B_1) = 0 \quad (\text{Eq. 3.10})$$

$$\hat{n} \times (E_2 - E_1) = 0, \quad \hat{n} \times (H_2 - H_1) = 0 \quad (\text{Eq. 3.11})$$

$$\text{Snell: } n_1 \cdot \sin\theta_1 = n_2 \cdot \sin\theta_2 \quad (\text{Eq. 3.12})$$

**Fresnel coefficients (dielectric-dielectric,  $\mu \approx \mu_0$ ).**

$$r_s = (n_1 \cos\theta_1 - n_2 \cos\theta_2) / (n_1 \cos\theta_1 + n_2 \cos\theta_2) \quad (\text{Eq. 3.13})$$

$$t_s = 2 n_1 \cos\theta_1 / (n_1 \cos\theta_1 + n_2 \cos\theta_2) \quad (\text{Eq. 3.14})$$

$$r_p = (n_2 \cos\theta_1 - n_1 \cos\theta_2) / (n_2 \cos\theta_1 + n_1 \cos\theta_2) \quad (\text{Eq. 3.15})$$

$$t_p = 2 n_1 \cos\theta_1 / (n_2 \cos\theta_1 + n_1 \cos\theta_2) \quad (\text{Eq. 3.16})$$

$$\text{Brewster (p-pol): } \tan \theta_B = n_2 / n_1 \quad (\text{Eq. 3.17})$$

$$\text{Critical angle (internal): } \theta_c = \arcsin(n_2 / n_1) \quad (\text{defined only if } n_1 > n_2) \\ (\text{Eq. 3.18})$$

### **Near- vs far-field (diffraction): Fresnel number.**

$$NF = a^2 / (\lambda \cdot z) \quad (\text{Fresnel if } NF \gtrsim 1; \text{ Fraunhofer if } NF \ll 1) \quad (\text{Eq. 3.19})$$

### **Replication (air → glass).**

Snell:  $\theta_2 \approx 28.13^\circ$  for  $\theta_1 = 45.0^\circ$ .

s-pol amplitudes:  $r_s \approx -0.303$  ( $R_s \approx 0.092$ ),  $t_s \approx 0.697$

p-pol amplitudes:  $r_p \approx 0.092$  ( $R_p \approx 0.008$ ),  $t_p \approx 0.728$

Brewster angle:  $\theta_B \approx 56.3^\circ$ .

Critical angle for glass → air ( $n_1=1.50$ ,  $n_2 \approx 1.0003$ ):  $\theta_c \approx 41.8^\circ$ .

### **Validity & limits.**

WKB/eikonal:  $|\nabla n|/n \ll k_0$  (index varies slowly on  $\lambda$  scale). Interface: locally planar, time-harmonic plane waves; no  $\sigma$ s or  $K$ s. Diffraction regimes per NF.

### **-- Interference — Two-Beam, Multi-Beam, Thin Films, Michelson --**

**Scope.** Interference only: superposition of coherent beams without invoking diffraction envelopes. We derive two-beam intensity, fringe conditions, visibility, multi-beam array factor, thin-film conditions (with reflection phase), and a Michelson rule of thumb. All formulas are plain text.

### **Two-beam interference (scalar, quasimonochromatic).**

$$E_1(t) = E_0 \cdot \cos(\omega t) \quad (\text{Eq. 5.1})$$

$$E_2(t) = E_0' \cdot \cos(\omega t + \Delta\varphi) \quad (\text{Eq. 5.2})$$

Time-averaged intensity (add irradiances plus cross term):

$$I = I_1 + I_2 + 2 \cdot \sqrt{I_1 \cdot I_2} \cdot \cos(\Delta\varphi) \quad (\text{Eq. 5.3})$$

Equal beams  $I_1 = I_2 = I_0 \Rightarrow I = 2 I_0 \cdot [1 + \cos(\Delta\varphi)]$  (max =  $4 I_0$ , min = 0).

Phase difference:  $\Delta\phi = (2\pi/\lambda) \cdot \Delta$  with  $\Delta =$  optical path difference (Eq. 5.4)

Bright fringes (no extra phase flips):  $\Delta = m \cdot \lambda$  ( $m = 0, \pm 1, \pm 2, \dots$ ) (Eq. 5.5)

Dark fringes:  $\Delta = (m + 1/2) \cdot \lambda$  (Eq. 5.6)

Fringe geometry for two equal, narrow slits separated by  $d$ , screen distance  $L$  (small angles):

Fringe angle spacing:  $\Delta\theta \approx \lambda / d$  (Eq. 5.7)

Fringe spacing on screen:  $\Delta y \approx (\lambda \cdot L) / d$  (Eq. 5.8)

### Visibility / contrast and coherence.

$V = (I_{\max} - I_{\min}) / (I_{\max} + I_{\min})$  (Eq. 5.9)

$V = [2 \cdot \sqrt{I_1 \cdot I_2} / (I_1 + I_2)] \cdot |\gamma_{12}(\tau)|$  (Eq. 5.10)

Here  $\gamma_{12}(\tau)$  is the complex degree of coherence at delay  $\tau$ ; for equal beams and full temporal coherence,  $V = 1$ .

### Multi-beam (N sources, equal spacing d, equal amplitudes).

Array phase:  $\delta = (2\pi/\lambda) \cdot d \cdot \sin\theta$  (Eq. 5.11)

Intensity:  $I(\theta) = I_0 \cdot [\sin(N \cdot \delta / 2) / \sin(\delta / 2)]^2$  (Eq. 5.12)

Principal maxima when  $\delta = 2\pi m \Rightarrow \sin\theta_m = m \cdot \lambda / d$ . Side-lobe structure set by  $N$ .

### Thin-film interference (normal incidence; include reflection phase).

Geometric path difference:  $\Delta = 2 \cdot n \cdot t$  (Eq. 5.13)

Phase on reflection (interface rule): a  $\pi$  phase flip occurs when reflecting from a lower-to-higher index boundary; no flip for higher-to-lower.

Cases (use the rule above to count  $\pi$  flips in the reflected paths):

- Case A — one  $\pi$  flip (e.g.,  $n_0 < n_1$  and  $n_1 > n_2$ , or  $n_0 > n_1$  and  $n_1 < n_2$ ):

Reflected constructive:  $2 \cdot n \cdot t = (m + 1/2) \cdot \lambda$

Reflected destructive:  $2 \cdot n \cdot t = m \cdot \lambda$

- Case B — two  $\pi$  flips (common AR stack with  $n_0 < n_1 < n_2$ ):

$$\text{Reflected constructive: } 2 \cdot n \cdot t = m \cdot \lambda$$

$$\text{Reflected destructive: } 2 \cdot n \cdot t = (m + 1/2) \cdot \lambda$$

- Case C — zero  $\pi$  flips (e.g.,  $n_0 > n_1 > n_2$ ): same conditions as Case B.

Oblique incidence:

$$\Delta = 2 \cdot n \cdot t \cdot \cos \theta_t \quad (\text{Snell's law for } \theta_t)$$

Quick replication (quarter-wave AR):

For two  $\pi$  flips (Case B), destructive reflection at first order occurs at  $m = 0 \rightarrow 2 \cdot n \cdot t = \lambda/2$ , so  $t = \lambda/(4n)$ . Example A:  $\lambda = 550 \text{ nm}$ ,  $n_{\text{film}} = 1.50 \rightarrow t \approx 91.7 \text{ nm}$ . Example B (typical AR material):  $\lambda = 550 \text{ nm}$ ,  $n_{\text{film}} = 1.38 \rightarrow t \approx 99.6 \text{ nm}$ .

Cross-link: reflection phase flips are the same Fresnel interface rules you derived in §3 (Interfaces).

### **Michelson interferometer (fringe count vs path change).**

$$\text{Optical path difference: } \Delta = 2 \cdot \Delta L \quad (\text{Eq. 5.17})$$

$$\text{One fringe shift when: } \Delta L = \lambda / 2 \quad (\text{Eq. 5.18})$$

Rule of thumb: moving one mirror by  $\lambda/2$  advances the fringe order by one.

### **Replication (quick numbers).**

Two-slit:  $\lambda = 632.8 \text{ nm}$ ,  $d = 300 \text{ }\mu\text{m}$ ,  $L = 1.5 \text{ m} \Rightarrow \Delta y \approx 3.16 \text{ mm}$ .

Thin film (reflection, one  $\pi$  flip):  $\lambda = 550 \text{ nm}$ ,  $n = 1.50 \Rightarrow$  first constructive at  $t = \lambda/(4n) \approx 91.7 \text{ nm}$ .

Michelson:  $\lambda = 500 \text{ nm} \Rightarrow$  one fringe per  $\Delta L = \lambda/2 = 250 \text{ nm}$  mirror motion.

### **Validity & limits.**

Assumes scalar, monochromatic or narrowband fields; paraxial geometry for fringe spacing; equal path polarization; ignores diffraction envelopes and vector corrections.

#### 4. Diffraction $\grave{a}$ — Kirchhoff, Fresnel, Fraunhofer

**Setup & scope.** Start from the scalar Helmholtz model, build the Kirchhoff integral via the free-space Green function, take the Fresnel and Fraunhofer limits, and list canonical patterns (single slit, double slit, grating, circular aperture). All formulas are plain text for easy reading.

##### **Helmholtz equation and free-space Green function.**

$$\nabla^2 U + k^2 \cdot U = 0 \quad \text{with} \quad k = 2\pi / \lambda \quad (\text{Eq. 4.1})$$

$$G(P,Q) = \exp(i k R) / (4 \pi R) \quad \text{with} \quad R = |P - Q| \quad (\text{Eq. 4.2})$$

##### **Kirchhoff integral (aperture $\Sigma$ to observation point P).**

$$U(P) = \iint_{\Sigma} [ U(Q) \cdot \partial G / \partial n - G \cdot \partial U / \partial n ] d\Sigma \quad (\text{Eq. 4.3})$$

##### **Fresnel approximation (paraxial).**

Geometry: aperture coordinates  $(\xi, \eta)$ , screen point  $(x, y)$ , separation  $z$ , small angles ( $\cos\theta \approx 1$ ). Use the quadratic path expansion:

$$R \approx z + [ (x - \xi)^2 + (y - \eta)^2 ] / (2 z) \quad (\text{Eq. 4.4})$$

Insert into the Kirchhoff integral and retain quadratic phase:

$$U(x,y; z) \approx [ \exp(i k z) / (i \cdot \lambda \cdot z) ] \iint U(\xi,\eta; 0) \cdot \exp\{ i k [ (x - \xi)^2 + (y - \eta)^2 ] / (2 z) \} d\xi d\eta \quad (\text{Eq. 4.5})$$

##### **Fraunhofer (far-field) approximation.**

For large  $z$  (or small aperture) so that the quadratic terms from  $\xi,\eta$  can be dropped inside the phase (keep only linear terms):

$$U(x,y; z) \approx [ \exp(i k z) / (i \cdot \lambda \cdot z) ] \cdot \iint U(\xi,\eta; 0) \cdot \exp\{ - i 2\pi [ \xi \cdot x + \eta \cdot y ] / (\lambda z) \} d\xi d\eta \quad (\text{Eq. 4.6})$$

Interpretation: up to the prefactor, the far-field is the 2-D Fourier transform of the aperture.

##### **Canonical intensity patterns.**

Single slit (width  $a$ , along  $x$ ; observe in  $y$ - $z$  plane):

$$I(\theta) = I_0 \cdot ( \sin \beta / \beta )^2 \quad \text{with} \quad \beta = (\pi a / \lambda) \cdot \sin\theta \quad (\text{Eq. 4.7})$$

Minima at  $a \cdot \sin\theta = m \cdot \lambda$  ( $m = \pm 1, \pm 2, \dots$ ).

Double slit (slit width  $a$ , center spacing  $d$ ):

$$I(\theta) = I_0 \cdot \cos^2(\pi d \sin\theta / \lambda) \cdot (\sin \beta / \beta)^2 \quad \text{with} \quad \beta = (\pi a / \lambda) \cdot \sin\theta$$

(Eq. 4.8)

Fringe spacing on a screen at distance  $z$  (small angles):  $\Delta y \approx \lambda z / d$ .

Transmission grating (period  $d$ ):

$$\text{Grating equation: } m \cdot \lambda = d \cdot \sin\theta_m \quad (m = 0, \pm 1, \pm 2, \dots) \quad (\text{Eq. 4.9})$$

Circular aperture (diameter  $D$ ): Airy pattern

$$I(\theta) = I_0 \cdot [2 J_1(\pi D \sin\theta / \lambda) / (\pi D \sin\theta / \lambda)]^2 \quad (\text{Eq. 4.10})$$

First minimum at  $\sin\theta \approx 1.22 \cdot \lambda / D$  (small-angle).

### **Fresnel zones (on-axis).**

$$\text{Zone radius: } r_n \approx \sqrt{(n \cdot \lambda \cdot z)} \quad (\text{Eq. 4.11})$$

Alternating zones add with alternating phase; a circular aperture of radius  $r_N$  passes about half the first zone amplitude when  $N \approx 1$ .

### **Regimes via Fresnel number.**

$$NF = a^2 / (\lambda \cdot z) \rightarrow \text{Fresnel if } NF \gtrsim 1; \text{ Fraunhofer if } NF \ll 1 \quad (\text{Eq. 4.12})$$

### **Replication (numbers you can check quickly).**

Single slit ( $\lambda=632.8$  nm,  $a=100$   $\mu\text{m}$ ,  $z=1.5$  m):

First minimum:  $\theta \approx \lambda/a = 0.00633$  rad  $\rightarrow y \approx 9.49$  mm on screen.

Fresnel number:  $NF = a^2/(\lambda z) \approx 0.01$  (near-intermediate).

Double slit (same  $\lambda, z$ ; spacing  $d=300$   $\mu\text{m}$ ):

Fringe spacing:  $\Delta y = \lambda z / d \approx 3.16$  mm.

Circular aperture ( $D=5$  mm, same  $\lambda$ ;  $z=2.0$  m):

Airy first minimum:  $\theta \approx 1.22 \lambda/D = 1.54e-04$  rad  $\rightarrow$  radius on screen  $y \approx 0.31$  mm.

### **Validity & limits.**

Scalar model with uniform polarization; paraxial angles for Fresnel/Fraunhofer forms; apertures large vs  $\lambda$  for Kirchhoff approximation; neglects vector/evanescent and edge polarization corrections.

## 5. Polarization Transport (Jones / Stokes / Mueller)

**Scope.** Model polarization with Jones vectors (amplitudes and phases), Stokes vectors (intensities), and Mueller matrices (system action). Include rotation (Faraday/optical activity), linear birefringence (retarders), diattenuation (polarizers), conversions, and quick numeric checks. All formulas are plain text for easy reading.

### Jones calculus (fully polarized fields).

$$\text{Jones vector: } E = [ E_x ; E_y ] \quad (\text{Eq. 5.1})$$

$$\text{Linear system (2}\times\text{2): } E_{\text{out}} = J \cdot E_{\text{in}} \quad (\text{Eq. 5.2})$$

$$\text{Rotation matrix: } R(\theta) = [ [ \cos\theta , -\sin\theta ] ; [ \sin\theta , \cos\theta ] ] \quad (\text{Eq. 5.3})$$

$$\text{Linear polarizer at angle } \theta: J_{\text{pol}}(\theta) = R(-\theta) \cdot [ [ 1, 0 ] ; [ 0, 0 ] ] \cdot R(\theta) \quad (\text{Eq. 5.4})$$

$$\text{Linear retarder (fast axis x, retardance } \delta): J_{\text{ret}}(\delta) = [ [ 1, 0 ] ; [ 0, \exp(i \cdot \delta) ] ] \quad (\text{global phase irrelevant}) \quad (\text{Eq. 5.5})$$

### Stokes vector and conversions (intensity domain).

$$\text{Stokes vector: } S = ( S_0 , S_1 , S_2 , S_3 ) \quad (\text{Eq. 5.6})$$

$$\text{From Jones (fully polarized): } S_0 = |E_x|^2 + |E_y|^2 \quad (\text{Eq. 5.7})$$

$$S_1 = |E_x|^2 - |E_y|^2 \quad (\text{Eq. 5.8})$$

$$S_2 = 2 \cdot \text{Re}( E_x \cdot E_y^* ) \quad (\text{Eq. 5.9})$$

$$S_3 = 2 \cdot \text{Im}( E_x \cdot E_y^* ) \quad (\text{Eq. 5.10})$$

$$\text{Degree of polarization: } P = \text{sqrt}( S_1^2 + S_2^2 + S_3^2 ) / S_0 \quad (\text{Eq. 5.11})$$

### Mueller matrices (act on Stokes).

$$S_{\text{out}} = M \cdot S_{\text{in}} \quad (\text{Eq. 5.12})$$

$$\text{Rotation by } \theta \text{ (e.g., Faraday/optical activity): } \quad (\text{Eq. 5.13})$$

$$S_{\text{out}} = (S_0, S_1 \cos 2\theta + S_2 \sin 2\theta, -S_1 \sin 2\theta + S_2 \cos 2\theta, S_3) \quad (\text{Eq. 5.14})$$

Linear polarizer at angle  $\theta$ : (Eq. 5.15)

$$M_{\text{pol}}(\theta) = (1/2) \cdot [ [ 1, \cos 2\theta, \sin 2\theta, 0 ]; \quad (\text{Eq. 5.16})$$

$$[ \cos 2\theta, \cos^2 2\theta, \sin 2\theta \cos 2\theta, 0 ]; \quad (\text{Eq. 5.17})$$

$$[ \sin 2\theta, \sin 2\theta \cos 2\theta, \sin^2 2\theta, 0 ]; \quad (\text{Eq. 5.18})$$

$$[ 0, 0, 0, 0 ] ] \quad (\text{Eq. 5.19})$$

Linear retarder (fast axis x, retardance  $\delta$ ): (Eq. 5.20)

$$M_{\text{ret}}(0, \delta) = [ [ 1, 0, 0, 0 ]; [ 0, 1, 0, 0 ]; [ 0, 0, \cos \delta, \sin \delta ]; [ 0, 0, -\sin \delta, \cos \delta ] ] \quad (\text{Eq. 5.21})$$

$$\text{General axis } \varphi: M_{\text{ret}}(\varphi, \delta) = R_M(-2\varphi) \cdot M_{\text{ret}}(0, \delta) \cdot R_M(2\varphi) \quad (\text{Eq. 5.22})$$

$$\text{where } R_M(\psi) \text{ acts as: } (S_0, S_1, S_2, S_3) \rightarrow (S_0, S_1 \cos \psi + S_2 \sin \psi, -S_1 \sin \psi + S_2 \cos \psi, S_3) \quad (\text{Eq. 5.23})$$

### Physical effects and parameter links.

$$\text{Faraday rotation (magneto-optic): } \theta_F = V \cdot B \cdot L \quad (\text{Eq. 5.24})$$

$$\text{Linear birefringence (uniaxial): } \delta = 2 \pi \cdot (\Delta n) \cdot L / \lambda \quad (\text{Eq. 5.25})$$

$$\text{Optical activity (chiral media): } \theta = \alpha \cdot L \quad (\alpha: \text{rotation per unit length}) \quad (\text{Eq. 5.26})$$

Diattenuation (unequal transmission): modeled by  $M_{\text{pol}}(\theta)$  with finite extinction ratio (Eq. 5.27)

### Replication (numbers you can check).

Faraday rotation:  $V = 4 \times 10^{-5} \text{ rad} \cdot \text{T}^{-1} \cdot \text{m}^{-1}$ ,  $B = 0.5 \text{ T}$ ,  $L = 0.2 \text{ m} \Rightarrow \theta_F \approx 4.0 \times 10^{-6} \text{ rad}$  ( $\approx 0.00023^\circ$ ).

Linear birefringence:  $\Delta n = 1 \times 10^{-5}$ ,  $L = 1 \text{ cm}$ ,  $\lambda = 632.8 \text{ nm} \Rightarrow \delta \approx 0.99 \text{ rad}$  ( $\approx 56.9^\circ$ ).

Malus' law check: two ideal linear polarizers at relative angle  $\Delta \rightarrow I_{\text{out}} = I_{\text{in}} \cdot \cos^2(\Delta)$ . For  $\Delta = 30^\circ$ ,  $\cos^2 \Delta \approx 0.75$ .

## Validity & limits.

Jones: fully polarized, coherent beams. Stokes/Mueller: works with partial polarization and depolarization; assumes quasi-monochromatic fields. Linear media; small-signal for  $V$ ,  $\Delta n$ . Dispersion handled by  $\lambda$ -dependence of  $V$  and  $\Delta n$ .

## 6. Imaging (Paraxial / ABCD) and Invariants

**Scope.** Paraxial ray transfer (ABCD) with height-angle vectors, element matrices (free space, thin lens), imaging rules, magnification, and the Lagrange invariant. All formulas are plain text; we use angles in radians and heights in meters.

### ABCD core relation (single transverse plane).

$$\begin{bmatrix} x_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix} \quad (\text{Eq. 6.1})$$

$$\text{Determinant (uniform index): } A \cdot D - B \cdot C = 1 \quad (\text{Eq. 6.2})$$

### Element matrices (uniform refractive index).

$$\text{Free-space propagation over length } L: F(L) = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \quad (\text{Eq. 6.3})$$

$$\text{Thin lens of focal length } f: L(f) = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \quad (\text{Eq. 6.4})$$

System matrix is the ordered product of elements (rightmost acts first).

$$\text{System: } M = F(L_2) \cdot L(f) \cdot F(L_1) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad (\text{Eq. 6.5})$$

### Imaging rules (thin lens in air).

$$\text{Object distance: } d_o \quad (\text{measured from lens}) \quad (\text{Eq. 6.6})$$

$$\text{Image distance: } d_i \quad (\text{measured from lens}) \quad (\text{Eq. 6.7})$$

$$\text{Lens equation: } 1/f = 1/d_o + 1/d_i \quad (\text{Eq. 6.8})$$

$$\text{Magnification: } M = -d_i / d_o \quad (\text{Eq. 6.9})$$

Connection to ABCD: an object at distance  $d_o$  before the lens and a screen at distance  $d_i$  after the lens yields a system matrix with  $C = -1/f + 1/d_i + 1/d_o = 0$  at focus. Equivalently, choose  $L_1 = d_o$ ,  $L_2 = d_i$  in  $M = F(L_2) \cdot L(f) \cdot F(L_1)$  and solve  $C = 0$  to recover  $1/f = 1/d_o + 1/d_i$ .

### Lagrange (optical) invariant in this angle-height convention.

$$\text{Invariant (no stops, lossless): } H = n \cdot x \cdot \sin\theta \approx n \cdot x \cdot \theta \quad (\text{small angles}) \quad (\text{Eq. 6.10})$$

H is conserved across lossless, paraxial systems; it bounds simultaneous concentration of height and angle (etendue).

### Replication (quick checks).

Thin-lens imaging (given):  $f = 50 \text{ mm}$ ,  $d_o = 100 \text{ mm} \Rightarrow d_i = 100 \text{ mm}$ ,  $M = -1$ .

ABCD multiplication example ( $d_o = 0.10 \text{ m}$ ,  $f = 0.050 \text{ m}$ ,  $d_i = 0.10 \text{ m}$ ).

$$F(L1) = \begin{bmatrix} 1 & 0.10 \\ 0 & 1 \end{bmatrix}, \quad L(f) = \begin{bmatrix} 1 & 0 \\ -20 & 1 \end{bmatrix}, \quad F(L2) = \begin{bmatrix} 1 & 0.10 \\ 0 & 1 \end{bmatrix}$$

$$M = F(L2) \cdot L(f) \cdot F(L1) = \begin{bmatrix} 0 & 0.10 \\ -10 & 0 \end{bmatrix}$$

Apply to a ray from an on-axis object point ( $x_1 = 0$ , small angle  $\theta_1$ ):  $[x_2; \theta_2] = [0, 0.10]; [-10, 0] \cdot [0; \theta_1] = [0.10 \theta_1; 0]$ . Focus at the image plane (angle goes to 0).

### Validity & limits.

Paraxial (small angles), thin elements, uniform index for the given determinant form. For systems with refractive interfaces (changing  $n$ ), use  $n$ -normalized slope  $q = n \cdot \theta$  so that  $\det = 1$  and the generalized Lagrange invariant  $H = x \cdot q$  is conserved.

## 7. Spectroscopy Anchors (Vacuum) and Ratio-First Validation

**Scope.** Vacuum hydrogen series from the Rydberg constant, ratio-first checks (insensitive to absolute calibration), and the tie-back to  $S_0 = \hbar$  (no new dimensional scales). All formulas are plain text.

### Rydberg constant (in terms of fundamental constants).

$$R_\infty = (\alpha^2 \cdot m_e \cdot c) / (2 \cdot h) \quad (\text{Eq. 7.1})$$

### Hydrogen wavelengths (vacuum):

$$1/\lambda = R_\infty \cdot (1/n_1^2 - 1/n_2^2) \quad \text{with} \quad n_2 > n_1 \quad (\text{Eq. 7.2})$$

Balmer examples ( $n_1 = 2$ ):  $H\alpha$  ( $3 \rightarrow 2$ ),  $H\beta$  ( $4 \rightarrow 2$ ),  $H\gamma$  ( $5 \rightarrow 2$ ).

### Ratio-first validation (insensitive to $R_\infty$ ):

$$\lambda_\alpha / \lambda_\beta = [ (1/2^2 - 1/3^2) ]^{-1} / [ (1/2^2 - 1/4^2) ]^{-1} = (36/5) / (16/3) = 27/20 \approx 1.35 \quad (\text{Eq. 7.3})$$

Measured (vacuum):  $656.281 \text{ nm} / 486.133 \text{ nm} \approx 1.350003 \rightarrow$  agreement within  $\sim 2 \text{ ppm}$ .

### Tie to $S_0 = \hbar$ (no new dimensional scales):

$$E_n = - (m_e \cdot e^2)^2 / (2 \cdot (4 \pi \epsilon_0)^2 \cdot \hbar^2 \cdot n^2) \quad (\text{Eq. 7.4})$$

$$a_0 = 4 \pi \epsilon_0 \cdot \hbar^2 / (m_e \cdot e^2) \quad (\text{Eq. 7.5})$$

These recover the Rydberg relations and depend only on  $\hbar$  imported at the electron anchor; no new scales are introduced in this pillar.

### **Validity & limits.**

Vacuum lines; non-relativistic hydrogen; reduced-mass correction small; QED/fine-structure and Lamb-shift corrections handled at higher order in the calibration capsule.

## **8. Small Corrections — Torsion / Shear and Closure Tolerance**

**Scope.** Parameterize small departures from the ideal vacuum transport via torsion and shear budgets, and relate experimental bounds to closure tolerance  $J_c$ .

### **Phase and index perturbations from torsion/shear budgets (first order).**

$$\delta\phi \approx \oint [\alpha_T \cdot T + \alpha_S \cdot S] \cdot dl \quad (\text{Eq. 8.1})$$

$$\delta n / n \approx \beta_T \cdot T + \beta_S \cdot S \quad (\text{Eq. 8.2})$$

Interpretation.

T and S are dimensionless small budgets imported from the Foundation (torsion and shear). Coefficients  $\alpha_T$ ,  $\alpha_S$ ,  $\beta_T$ ,  $\beta_S$  are geometry-dependent but order-one in calibrated units. In pure vacuum these corrections are expected to vanish; any measured residual constrains  $J_c$ .

### **Closure tolerance link.**

$$J_c = \Delta S / S_0 \quad (\text{Eq. 8.3})$$

Use precision optics bounds (e.g.,  $\delta n/n \lesssim 10^{-6}$ ) to set  $|T|, |S| \lesssim 10^{-6}$  (up to  $\beta$ -coefficients). These feed back into error budgets for propagation and spectroscopy.

### **Validity & limits.**

Applies when  $|T|, |S| \ll 1$  (near-ideal vacuum). Odd azimuthal modes are parity-suppressed unless explicit chirality is present.

## 9. Falsifiers & Lab-Math Hooks (Neutral Rotor)

**Scope.** Neutral, non-magnetic rotor aligned to the global caustic plane. Predict a tiny, orientation-gated push/zero/pull with spin flips, then show how to bound it if absent. All formulas are plain text.

### Setup and definitions.

Composite body is neutral:  $\Sigma q = 0$  (no net monopole). Use a dielectric rotor (e.g., fused silica), non-magnetic bearings, high-vacuum, and electrostatic shielding. Align the spin axis with the caustic-plane normal.

$$\varepsilon = \text{closure / tear bias (small, dimensionless)} \quad (\text{Eq. 9.1})$$

$$\sigma = \text{spin orientation ( +1 for right-handed / with } +\hat{n}, -1 \text{ for opposite )} \\ (\text{Eq. 9.2})$$

$$\theta = \text{facing (tilt) angle relative to } +\hat{n} \quad (\theta = 0 \rightarrow \text{right facing; } \theta = 90^\circ \rightarrow \\ \text{null}) \quad (\text{Eq. 9.3})$$

### Prediction (cycle-averaged).

$$\langle \Delta p \rangle \propto \varepsilon \cdot \sigma \cdot \cos\theta \quad (\text{Eq. 9.4})$$

Signature (at fixed speed, no EM handles):

- right spin + right facing ( $\sigma = +1, \theta = 0^\circ$ )  $\rightarrow$  push
- $90^\circ$  tilt ( $\theta = 90^\circ$ )  $\rightarrow$  strict null
- spin-flip or opposite facing ( $\sigma = -1$  or  $\theta = 180^\circ$ )  $\rightarrow$  pull

### From momentum to force (lock-in readout).

$$F_{\text{sig}} \approx \Omega \cdot |\langle \Delta p \rangle| \quad (\text{Eq. 9.5})$$

Modulate the spin state ( $\sigma \rightarrow -\sigma$ ) or dither  $\theta$  about  $0^\circ$  by  $\pm\delta\theta$  at angular frequency  $\Omega$  and demodulate at  $\Omega$ . The signal flips sign with  $\sigma$  and follows  $\cos\theta$ .

### Replication (illustrative numbers).

Toy:  $\varepsilon = 1 \times 10^{-3}$ ,  $\sigma = +1$ ,  $\theta = 0^\circ$ ,  $p_0 = 1 \times 10^{-3}$  N·s,  $\Omega = 100$  Hz  $\Rightarrow |\langle \Delta p \rangle| \approx 1.0 \times 10^{-6}$  N·s,  $F_{\text{sig}} \approx 1.0 \times 10^{-4}$  N.

Micro-newton torsion balances can reach  $\sim 10^{-8}$ – $10^{-7}$  N; if null at that level, the bound is  $\varepsilon \lesssim F_{\text{min}} / (\Omega \cdot p_0 \cdot |\cos\theta|)$ .

$$\varepsilon_{\text{bound}} \lesssim F_{\text{min}} / (\Omega \cdot p_0 \cdot |\cos\theta|) \quad (\text{Eq. 9.6})$$

### Controls and nulls.

- Electrostatic: conductive coating + Faraday cage; discharge rotor; monitor surface potential.
- Magnetic: non-magnetic materials,  $\mu$ -metal shielding; verify  $\sigma$  flips do not change stray fields.
- Thermal/flow: high vacuum, constant temperature; check  $\sigma$ -synchronous heating.
- Geometry nulls:  $\theta = 90^\circ$  (strict null),  $\sigma$  flip (sign flip).

### Validity & interpretation.

Applies to neutral, non-magnetic, isolated rotors aligned to the global caustic plane. A non-zero  $\sigma$ -locked signal is new relative to standard EM; a null sets a quantitative upper bound via the formula above.

## 10. Replication Blocks (Bench-Ready Numbers)

### Single-slit diffraction (Fraunhofer).

$$\text{First minimum: } y_1 = (\lambda \cdot L)/a \quad (\text{Eq. 10.1})$$

He-Ne  $\lambda = 632.8 \text{ nm}$ ,  $a = 100 \text{ }\mu\text{m}$ ,  $L = 1.00 \text{ m} \Rightarrow y_1 \approx 6.33 \text{ mm}$ .

$$\text{Fresnel number: } N_F = a^2 / (\lambda \cdot L) \quad (\text{Eq. 10.2})$$

With the same numbers:  $N_F \approx 0.016$  ( $\ll 1 \rightarrow$  Fraunhofer valid).

### Double-slit interference (no envelope).

$$\text{Fringe spacing: } \Delta y = (\lambda \cdot L)/d \quad (\text{Eq. 10.3})$$

$\lambda = 632.8 \text{ nm}$ ,  $d = 300 \text{ }\mu\text{m}$ ,  $L = 1.00 \text{ m} \Rightarrow \Delta y \approx 2.109 \text{ mm}$ .

### Interfaces (air $\leftrightarrow$ glass).

$$\text{Brewster angle (air} \rightarrow \text{glass): } \tan \theta_B = n_2 / n_1 \quad (\text{Eq. 10.4})$$

$n_1 = 1.0003$  (air),  $n_2 = 1.50$  (glass)  $\Rightarrow \theta_B \approx 56.3^\circ$ .

$$\text{Critical angle (glass} \rightarrow \text{air): } \sin \theta_c = n_2 / n_1 \quad (\text{with } n_1 > n_2) \quad (\text{Eq. 10.5})$$

For glass  $\rightarrow$  air:  $\sin \theta_c = n_{\text{air}} / n_{\text{glass}} \Rightarrow \theta_c \approx 41.8^\circ$ .

Constants and data sources (for your overall document's references): CODATA 2022; NIST ASD v5.11 (2023) Balmer wavelengths; PDG muon lifetime unchanged (context only).

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